The Ultimate Flat Tire

The unusual road surface that is installed here is made up of mathematically precise arches, designed so that a square wheel will roll smoothly over the bumps. Try it for yourself, being careful before you start to line up the corners of the wheels in the low points of the road.

The Problem Suppose a bicycle has square wheels. How can a roadbed be designed so that the ride is smooth, in the sense that the seat of the bike stays horizontal?

History The problem was first considered and solved by G. B. Robison in 1960. In 1992 Leon Hall and Stan Wagon investigated other possibilities for shapes of wheels. The first square-wheel bike at Macalester was installed in 1998 and was featured in many television and newspaper pieces, including Ripley's *Believe It Or Not*. The new, improved model was installed in 2004.

What Is It Good For? Round pieces of wood (quarter-circles) have been found near the sites of Egyptian pyramids and it has been suggested that they were used as a road bed in roughly the shape as the one here, so that large blocks of stone could be moved efficiently.

The Answer Each arch of the road is an inverted catenary, the curve one gets by just dropping a piece of chain or rope while holding its ends.

Derivation of the Solution Intending to build a model of our solution (using a tricycle to improve stability), we first noted that a commercially available tricycle could be modified slightly so that the distance between its front and rear axles was exactly 42 inches. The roadbed would be a series of arches, and we decided to have three arches span the wheelbase, so that each arch would have a horizontal span of 14 inches.

For a coordinate system we let the horizontal line tangent to the top of these arches be the x-axis and we draw the y-axis through the top of the central arch.

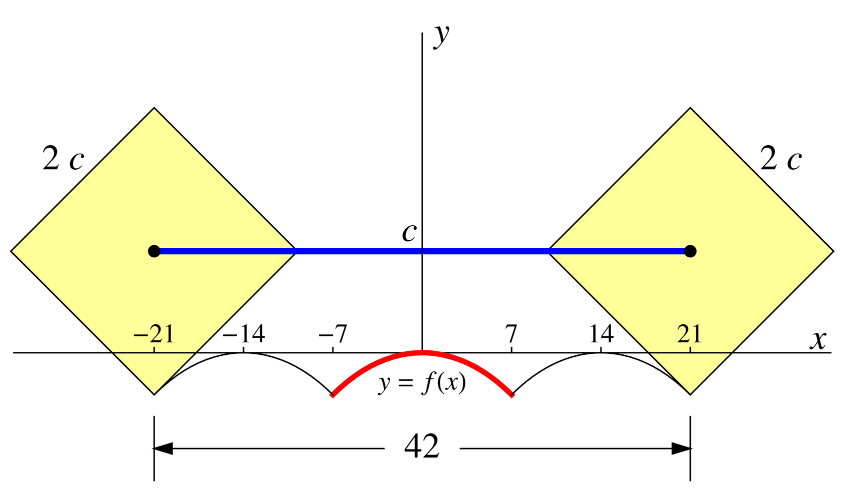

Figure 1. The problem is to determine the red curve so that the center of the square will stay on the blue line as the square rolls.

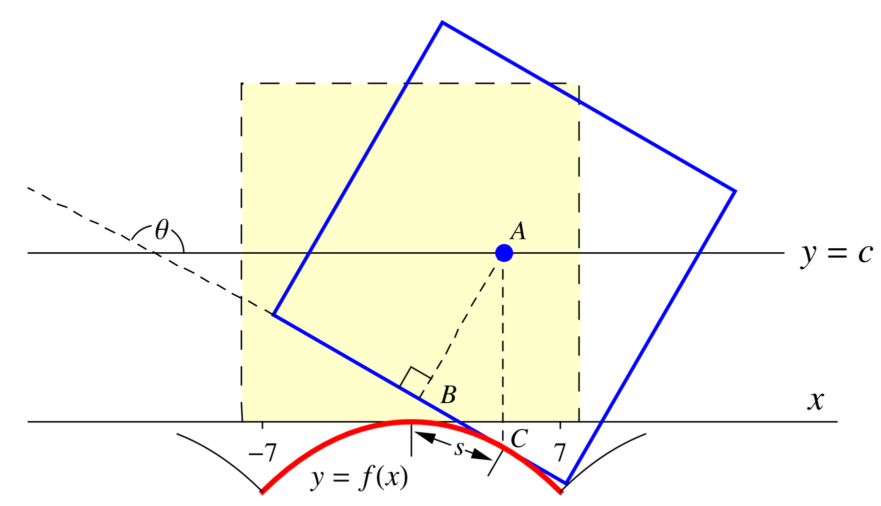

Figure 2. As the square rolls, the arc length along the curve, s, must match the amount of wheel that has come into contact with the curve, which is the line from B to C.

Our task is to determine $y = f(x)$, the equation of the arch (red in Fig. 1), on which the square will roll forward. Suppose the square's sides have length $2c$, so that when it sits on the x-axis, centered around the y-axis, the center of the square is at the point c on the y-axis. The rider is to experience a smooth ride, so the center of the square is to move along the line $y = c$ as the square rolls forward. Since one arch spans a horizontal distance of $\frac{1}{3}\,42$, or 14, the curve we seek goes from $x = -7$ to $x = 7$; and since a corner of the wheel should rest in the cusp where two arches join at $x = 7$, the slope of the curve at $x = 7$ must be exactly -1. In calculus terms, $f'(7) = -1$. Figure 2 shows the original position of the square in yellow and the position after it has rolled forward in blue.

Suppose the side of the square tangent to the curve at C makes an angle of θ with the x-axis; then $f'(x) = \tan\theta$. With A as the center of the wheel and B as the foot of the perpendicular dropped from A, $\angle BAC = \pi - \theta$, so

$$\tan\theta = -\tan(\pi - \theta) = -\tan(\angle BAC) = -\frac{|BC|}{c}.$$

Since the wheel rolls without slipping, $|BC|$ must equal s, the distance measured along the arch. From the arc length formula of calculus, this means

$$\frac{dy}{dx} = \tan\theta = -\frac{s}{c} = -\frac{1}{c}\int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Differentiating both sides gives $y'' = -\frac{1}{c}\sqrt{1 + (y')^2}$.

Let $z = y'$ and this becomes $\frac{dz}{dx} = -\frac{1}{c}\sqrt{1 + z^2}$, or $\frac{1}{\sqrt{1+z^2}}\, dz = -\frac{1}{c}\, dx$. The integral of the left side is the inverse hyperbolic sine, so $\sinh^{-1} z = -\frac{1}{c}x + k$, which means that $z = \sinh\left(-\frac{1}{c}x + k\right)$.

Since $z = dy/dx = 0$ when $x = 0$, k vanishes and $z = \sinh(-x/c) = -\sinh(x/c)$. Integrating once more then gives $y = (-c\cosh(x/c)) + K$. But y vanishes when $x = 0$, so $K = c$, and we also want $y' = -1$ when $x = 7$. This means $-1 = \sinh(7/c) = \frac{1}{2}\left(e^{7/c} - e^{-7/c}\right)$, which can be solved to give $c = 7/\operatorname{arcsinh} 1 = 7.942$. So we finally have the desired curve:

$$y = 7.942 - 7.942\cosh\frac{x}{7.942}.$$

We recognize this as the equation of an upside-down catenary, the shape made by a chain hanging from two ends. The side length of the square wheel that will roll smoothly on a roadbed of such catenaries is therefore $2c$, or 15.884 inches. The chain connection makes it easy to make a pattern for constructing the road: just suspend a chain of length $15 \cdot \frac{7}{8}$ inches from two nails.

Comparisons to More Familiar Motions A point on a rolling wheel generates a curve called a *cycloid*. If the point is farther from the center than the rolling radius, as happens with a train wheel, then we see that the point travels backwards, even as the train travels forward. The locus of a point on the square wheel shares this property with train wheels: The corner of the square follows a path that involves backwards motion when it leaves the cusp. This can be seen in the backward motion of the red dots in Figure 3 just as the dot leaves the cusp.

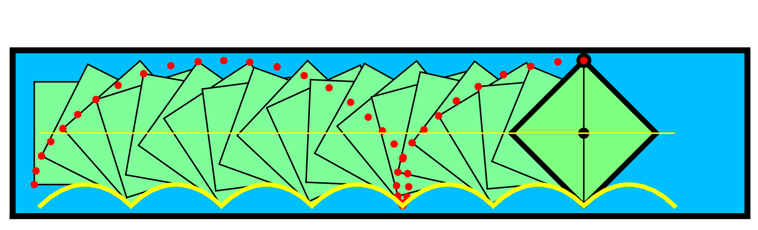

Figure 3. As the square rolls, the corner of the square traces out a curve with a loop, which means that as it leaves the cusp, the corner of the square travels backward.

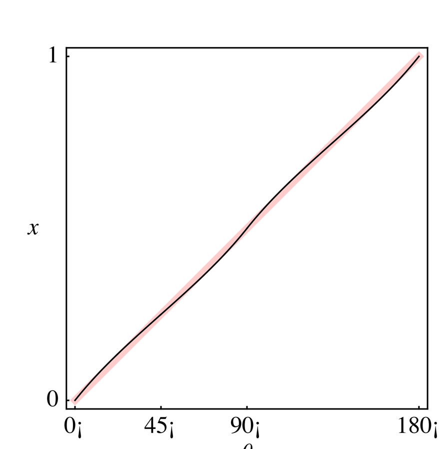

Figure 4. As θ increases, x, the distance traveled by the wheel, increases too, but it is slightly nonlinear, as shown by the black curve.The thick pink line shows the perfectly linear relationship for a round wheel.

For a round bicycle wheel there is a simple relationship between the speed of pedaling and the speed of motion. If one pedals twice as fast, then the bike goes twice as fast. In other words, the x vs. θ relationship is linear — $x = K\theta$ — with the constant K depending on the size of the wheel and the gearing. Things are different for a square wheel: the relationship is not linear (Fig. 4). However, the departure from linearity is quite small. The function that describes it for the bike on display here (ignoring constants) is $x = \operatorname{arctanh}(\tan\theta)$.

Any shape can be used as a wheel, but some work better than others. For example, it is impossible to build a contraption with triangular wheels. The reader might enjoy puzzling out why this is the case.

Enjoy your ride, as did Macalester President Brian Rosenberg in the inaugural ride on February 25, 2004.

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