May 2016

Round Formulas for Exponential Polynomials and Gamma

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Abstract. The nth partial sum of the Maclaurin series for $e^{a/b}$, where a and b are integers, becomes an integer when multiplied by $n!\, b^n$. This integer is related to many combinatorial properties of interest and is also directly tied to an exact computation of $\Gamma\left(n, \frac{a}{b}\right)$, where Γ is the incomplete gamma function. This paper presents very short formulas that give this integer exactly when $|a| \le 2$. For larger a the method extends and while not as fast as the smaller cases, it is an improvement on existing computational methods. The cases $a = \pm 1$ were known for many b-values. The approach here extends the general idea to all rationals by making use of a congruence to overcome the error inherent in the truncation of the Maclaurin series. A side-effect of the investigation is a new analytic lower bound on the number of times a prime a appears in a factorial: $\frac{n}{a-1} - \log_a(n+1)$.

Introduction

Let $e_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}$ be the partial sum for the Maclaurin series of e^z; $\{\cdot\}$ is the rounding function; a and b always denote integers, with $b \ge 0$ and $\gcd(a, b) = 1$; n denotes a nonnegative integer. Clearly $b^n\, n!\, e_n\left(\frac{a}{b}\right)$ is an integer—call it $K_n\left(\frac{a}{b}\right)$, often abbreviated to just K—since all denominators in the sum are cleared. Here we investigate short and efficient formulas that give K's exact value. Table 1 shows several of these sequences together with their location in the On-Line Encyclopedia of Integer Sequences [1].

The main result here is that very simple "Round Formulas" for K exist when $a = \pm 1$ or ± 2, b is any nonzero integer, and n is any positive integer. When the numerator is ± 1, the formula is remarkably simple: $\left\{b^n\, n!\, e^{\pm 1/b}\right\}$; for $a = \pm 2$ it is a little more complicated but can still be called a one-liner. In these cases the computational load is the single high-precision evaluation of the rational power of e . There are extensions to $a = \pm 3$ and greater, but their improvement over the method of brute-force addition is modest: an n-step algorithm is reduced to one using about $\frac{n}{\log_a n}$ steps. Here the computational load becomes the evaluation of a large congruential residue.

Many special cases when $a = \pm 1$ were known (e.g., $K_n\left(\frac{-1}{2}\right)$ by S. Plouffe [3] and M. van Hoeij [1, A000354]), but we here provide a general method for $K_n\left(\frac{\pm 1}{b}\right)$ as well as a similarly efficient formula for $K_n\left(\frac{\pm 2}{b}\right)$. By the classic identity $n!\, e_n(z) = e^z\, \Gamma(n+1, z)$, where Γ is the incomplete gamma function (defined as $\Gamma(n, z) = \int_z^\infty e^{-t}\, t^{n-1}\, dt$), any formula for $K_n\left(\frac{a}{b}\right)$ yields a similar formula that expresses $\Gamma\left(n+1, \frac{a}{b}\right)$ in symbolic form as $K_n\left(\frac{a}{b}\right) e^{-a/b}\, b^{-n}$.

a	b	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	OEIS catalog
0	1	1	1	2	6	24	120	720	A000142
1	1	1	2	5	16	65	326	1957	A000522
1	2	1	3	13	79	633	6331	75 973	A010844
1	3	1	4	25	226	2713	40 696	732 529	A010845
1	4	1	5	41	493	7889	157 781	3 786 745	A056545
1	5	1	6	61	916	18 321	458 026	13 740 781	A056546
1	6	1	7	85	1531	36 745	1 102 351	39 684 637	A056547
−1	1	1	0	1	2	9	44	265	A000166
−1	2	1	1	5	29	233	2329	27 949	A000354
−1	3	1	2	13	116	1393	20 894	376 093	A000180
−1	4	1	3	25	299	4785	95 699	2 296 777	A001907
−1	5	1	4	41	614	12 281	307 024	9 210 721	A001908
−1	6	1	5	61	1097	26 329	789 869	28 435 285	——
2	1	1	3	10	38	168	872	5296	A010842
2	3	1	5	34	314	3784	56 792	1 022 320	A097817
2	5	1	7	74	1118	22 376	559 432	16 783 024	A097821
−2	1	1	−1	2	−2	8	8	112	A000023
−2	3	1	1	10	82	1000	14 968	269 488	——
−2	5	1	3	34	502	10 056	251 368	7 541 104	——
3	1	1	4	17	78	393	2208	13 977	A053486
3	2	1	5	29	201	1689	17 133	206 325	——
3	4	1	7	65	807	12 993	260 103	6 243 201	——
3	5	1	8	89	1362	27 321	683 268	20 498 769	——
−3	1	1	−2	5	−12	33	−78	261	A010843
−3	2	1	−1	5	3	105	807	10 413	——
−3	4	1	1	17	177	2913	58 017	1 393 137	——
−3	5	1	2	29	408	8241	205 782	6 174 189	——

Table 1. $n!\,b^n\,e_n\left(\frac{a}{b}\right)$ for $|a| \le 3$, $1 \le b \le 6$, and $0 \le n \le 6$.

The integer K often has interesting combinatorial interpretations. For example, let $T(n)$ be the number of permutations of $\{1,\ldots,n\}$ having at least one transposition [1, A000266]. Let $T^c(n)$ count the permutations having no transposition: $T^c(n) = n! - T(n)$. Then, with $m = \left\lfloor \frac{n}{2} \right\rfloor$, $T^c(n) = n!\sum_{k=0}^{m}(-1)^k \frac{1}{2^k\,k!} = \frac{n!}{2^m\,m!} K_m\left(\frac{-1}{2}\right)$, where the first equality is a standard exercise using the principle of inclusion and exclusion [2]. Here is a sampling of some others (see the corresponding entries at [1], as listed in Table 1).

- $K_n(1)$ counts the total number of ordered tuples using distinct elements of $\{1, 2,\ldots, n\}$, where the tuple's length is from 0 to n, inclusive.

- $K_n(-1)$ gives the number of derangements of an n-element set: permutations with no fixed point (also known as subfactorial(n)).
- $K_n\left(\frac{1}{2}\right)$ is the number of ways to sort a spreadsheet with n columns.
- $K_n\left(\frac{-1}{2}\right)$ is the number of ways to choose a permutation of $\{1, 2, \ldots, n\}$ and choose k elements $(0 \le k \le n)$ that are not fixed points of the permutation.

Here are some general properties of K; we do not use these, but list them for completeness.

- Asymptotics: $K_n(z) \sim n!\, b^n\, e^z$. (This is because $e_n(z)$ is asymptotic to e^z.)
- Recurrence: $K_n\left(\frac{a}{b}\right) = b\, n\, K_{n-1}\left(\frac{a}{b}\right) + a^n$. (Proof: Follows directly from the sum in the definition.)
- Integral characterization: $K_n\left(\frac{a}{b}\right) = \Gamma\left(n+1,\ \frac{a}{b}\right) = e^{a/b}\, b^n \int_{a/b}^{\infty} e^{-t}\, t^n\, dt$. (Proof: Integration by parts shows that the integral satisfies the same recurrence as K.)
- The exponential generating function of the sequence $\left(K_n\left(\frac{a}{b}\right)\right)_{n\ge 0}$ is $f = \frac{e^{a x}}{1 - b x}$. (Proof: $\partial_x^n(f) = f\, \frac{n!\, b^n}{(1-b x)^n} \sum_{k=0}^{n} \frac{(1-b x)^k}{k!} \left(\frac{a}{b}\right)^k$; at $x = 0$ this is $n!\, b^n \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{a}{b}\right)^k$.)
- $K_n(a)$ is the permanent of the $n \times n$ matrix with $a+1$ at each diagonal entry and 1 in all other entries. (Proof: Use the definition of the permanent and classify the permutations according to how many fixed points each has.)

The starting point is a simple theorem showing that, under certain conditions, the integer K can be expressed by a very short formula.

Theorem 1. Suppose $a \in \mathbb{Z}$, $b, n \in \mathbb{N}$, $b \ge 1$, and $b\,(n+1) > |a|$, and $\frac{|a|^{n+1}}{b(n+1)-|a|} \le \frac{1}{2}$. Then

(Round Formula) $$K_n\left(\frac{a}{b}\right) = b^n\, n!\, e_n\left(\frac{a}{b}\right) = \left\{b^n\, n!\, e^{a/b}\right\}$$

Proof. The tail of the Maclaurin series for e^z after the nth term is $\sum_{j=1}^{\infty} \frac{z^{n+j}}{(n+j)!}$. Multiplication by $b^n n!$ gives $b^n \sum_{j=1}^{\infty} \frac{n!\, z^{n+j}}{(n+j)!}$, which is strictly less than $b^n \sum_{j=1}^{\infty} \frac{z^{n+j}}{(n+1)^j}$. This last, setting $z = a/b$, is a convergent geometric series with sum $\frac{\left|a^{n+1}\right|}{(n+1)\, b - |a|}$. Therefore $\left|b^n\, n!\, e^{a/b} - b^n\, n!\, e_n\left(\frac{a}{b}\right)\right| < \frac{|a|^{n+1}}{b(n+1)-|a|} \le \frac{1}{2}$. Because $b^n\, n!\, e_n\left(\frac{a}{b}\right)$ is an integer, the proof is complete. □

The Case $a = \pm 1$

When $a = \pm 1$, the condition of Theorem 1 holds in almost all cases. Note that initial cases $(n \le 2)$ are quite trivial: $K_0\left(\frac{a}{b}\right) = 1$, $K_1\left(\frac{a}{b}\right) = a + b$, and $K_2\left(\frac{a}{b}\right) (a+b)^2 + b^2$.

Corollary 2. Suppose $a = \pm 1$, n is a positive integer, $b \in \mathbb{N}$ with $b \ne 0$, and $(b, n) \ne (1, 1)$. Then $b^n\, n!\, e_n\left(\frac{a}{b}\right) = \left\{b^n\, n!\, e^{a/b}\right\}$ and $\Gamma\left(n+1,\ \frac{a}{b}\right) = \left\{b^n\, n!\, e^{a/b}\right\} b^{-n}\, e^{-a/b}$.

Proof. Use Theorem 1; because $a = \pm 1$, the restrictive inequality holds for all n. The excluded case means that either $n \ge 2$ or $b \ge 2$, and for such values the critical quantity $\frac{|a|^{n+1}}{b(n+1)-|a|} = \frac{1}{b\,(n+1)-1}$ is less than $\frac{1}{2}$. □

To compute $K_n\left(\frac{\pm 1}{b}\right)$ in *Mathematica* when $n \ge 2$, one just uses `Round[n ! b^n e^(a/b)]`. And $\Gamma_n\left(\frac{\pm 1}{b}\right)$ is given by `Round[n ! b^n e^(a/b)] b^(-n) e^(-a/b)`.

We can now compute instantaneously the exact number of permutations with no transposition, as first noted in [3] and [1, A000266, A000354].

Corollary 3. If $m = \lfloor \frac{n}{2} \rfloor$, then $T^c(n) = \frac{n!}{2^m\, m!} \left\{ \frac{2^m\, m!}{\sqrt{e}} \right\}$.

Proof. Use Corollary 2 with $a = -1$ and $b = 2$. Then

$$T(n) = n! \sum_{k=1}^{m} \frac{(-1)^{k+1}}{2^k\, k!}$$

$$= -n! \sum_{k=1}^{m} \left(-\frac{1}{2}\right)^k \frac{1}{k!} = n! - n! \sum_{k=0}^{m} \left(-\frac{1}{2}\right)^k \frac{1}{k!} = n! - \frac{n!}{m!\, 2^m} K_m\left(\frac{-1}{2}\right) = n! - \frac{n!}{2^m\, m!} \left\{ \frac{2^m\, m!}{\sqrt{e}} \right\}. \;\square$$

The Case $a = \pm 2$

Next we turn to the case $a = \pm 2$ with b odd. The integer K has certain arithmetic properties that will allow us to overcome the uncertainty caused by the Maclaurin series truncation. As in Theorem 1's proof, the Maclaurin error is bounded in absolute value by $\frac{|a|^{n+1}}{b\,(n+1)-|a|}$, or $\frac{2^{n+1}}{b\,(n+1)-2}$; the direction of the error is determined by the sign of a and the parity of n: it is negative iff $a < 0$ and n is even. So this determines an interval in which K lies; when $|a| = 1$, the interval has length at most $1/2$. For larger a the error can be much larger, but for $a = \pm 2$, we can appeal to a simple congruence condition on K to overcome the error.

When a is prime, we use $p_n(a)$ for the exponent of a in $n!$. We start by investigating upper and lower analytic bounds on $p_n(a)$. The proof of Lemma 4 when $a \geq 3$ was provided by Robert Israel (Univ. of British Columbia) and is included with his permission.

Lemma 4. If n is a positive integer and a is prime, then $\left\lceil \frac{n}{a-1} - \log_a(n+1) \right\rceil \leq p_n(a) \leq \left\lfloor \frac{n-1}{a-1} \right\rfloor$.

Proof. Let $D = \{d_j\}$ be the set of base-a digits of n; then $|D| = L$, where $L = \lfloor 1 + \log_a n \rfloor$ and $a^{L-1} \leq n < a^L$. Let $\sigma = \Sigma D$. Recall the Legendre formula [5] that $p_n(a) = \frac{n-\sigma}{a-1}$. This immediately gives the upper bound.

The lower bound follows from $\sigma \leq (a-1)\log_a(n+1)$, which is the same as $a^{\sigma/(a-1)} \leq n+1$. Define F, considered as a real-valued function of the d_j, to be $F(d_1, \ldots, d_L) = (a-1)\log_a\left(1 + \sum_{j=0}^{L-1} d_j\, a^j\right) - \sum_{j=0}^{L-1} d_j$. Then the inequality we seek is equivalent to $F(d_1, \ldots, d_L) \geq 0$. But F is a concave function and so its minimum must occur at an extreme point of $[0, a-1]^L$, i.e., at the vector where each entry is either 0 or $a-1$. Now, if such a vector has one or more entries equal to 0, then the claimed inequality holds because $a^{\sigma/(a-1)} \leq a^{L-1} \leq n$, which means that $F \geq 0$. And if none are 0 then all are $a-1$, which means $n = a^L - 1$; then $a^{\sigma/(a-1)} = a^L = n+1$, which again means $F \geq 0$. Hence $F \geq 0$ holds in all cases. $\square$

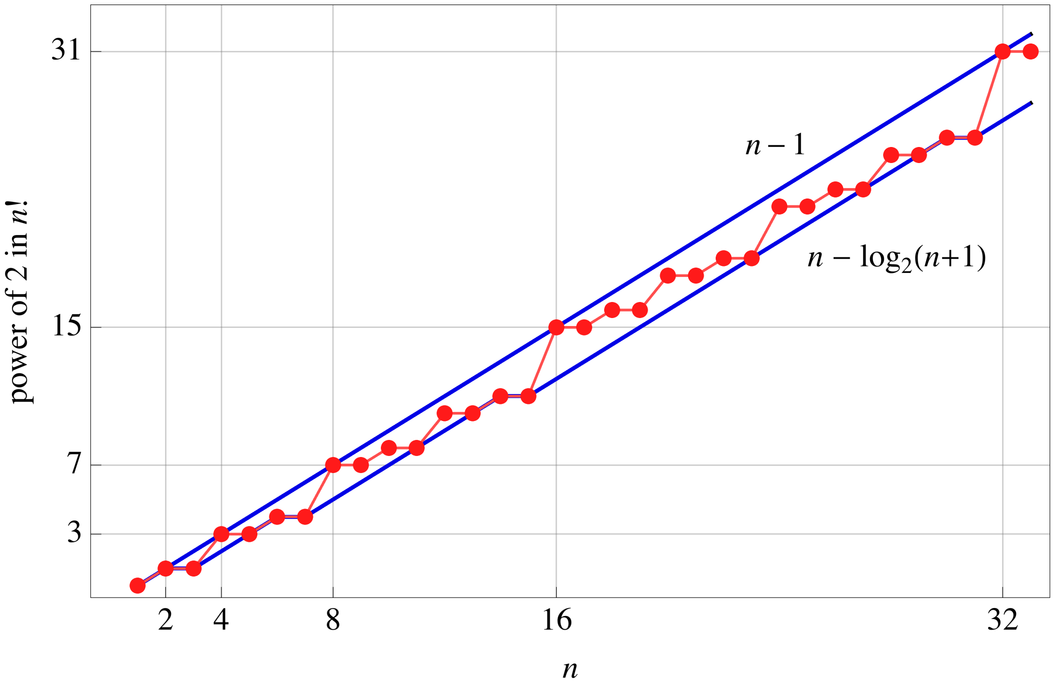

Figure 1. The bounds on the power of 2 in $n!$ in Lemma 4 are fairly sharp.

Next we find a universal congruence condition for $K_n\left(\frac{a}{b}\right)$.

Lemma 5. For $b \in \mathbb{N}$ and prime $a \in \mathbb{Z}$ with $\gcd(a, b) = 1$, let $p = p_n(a)$, $q = \frac{n!}{a^p}$, and $L = \operatorname{mod}(b^n q, |a|)$. Then $K_n\left(\frac{a}{b}\right) \equiv L\, a^p \left(\operatorname{mod} |a|^{p+1}\right)$; it follows that $K_n\left(\frac{a}{b}\right) \equiv 0 \;(\operatorname{mod} |a|^p)$.

Proof. We have $K_n\left(\frac{a}{b}\right) = \sum_{k=0}^{n} a^k\, b^{n-k}\, \frac{n!}{k!}$. The proof will proceed by showing that the first term in the sum is congruent to $L\, a^p \left(\operatorname{mod} |a|^{p+1}\right)$ while the others are each divisible by a^{p+1}. The first term is $b^n\, n! = b^n\, a^p\, q$. Then $b^n\, n! - L\, a^p = b^n\, q\, a^p - L\, a^p = a^p(b^n\, q - L)$, which is divisible by a^{p+1} by the definition of L; therefore $b^n\, n! \equiv L\, a^p \left(\operatorname{mod} |a|^{p+1}\right)$, as claimed. For the second result, it suffice to show that a^k has at least one more a in it than does $k!$. This follows from the upper bound of Lemma 4: $p_a(k) \le \frac{k-1}{a-1} \le k - 1$. □

Now we can use the two lemmas to derive an exact formula for $K_n\left(\frac{\pm 2}{b}\right)$. The main point is that the Maclaurin error is about 2^{n+1} while the simple modulus a^p from Lemma 4 is about $\frac{1}{n} 2^n$. For the modulus to exceed the error, we need to either find a larger modulus for a congruence condition or reduce the Maclaurin error. The latter approach works well, where we simply add in one more term to the partial sum to better estimate the truncation error.

Theorem 6 (The case $a = \pm 2$). Assume $a = \pm 2$ and $b, n \in \mathbb{N}$ with $n \ge 3$, b odd, and $(b, n) \ne (1, 3)$. Let $r = \left\{ b^n\, n!\, e^{a/b} - \frac{a^{n+1}}{(n+1)\, b} \right\}$, $m = 2^{\lceil n - \log_2(n+1) \rceil}$, and $s = \operatorname{sign}(a)^n$. Then $K_n\left(\frac{a}{b}\right) = r - s\, \operatorname{mod}_m(s\, r)$. And $\Gamma\left(n+1, \frac{a}{b}\right) = e^{-a/b}\, b^{-n}(r - s\, \operatorname{mod}_m(s\, r))$.

Proof. By Lemma 4, m is a lower bound on the power of 2 in $n!$; therefore $K_n\left(\frac{a}{b}\right) \equiv 0 \;(\operatorname{mod} m)$. The theorem starts with r and then adjusts it, in the proper direction, so that it becomes divisible by m. Note that the adjustment direction depends only on the sign of a and not on the parity of n. Now, the truncation error bound (see Thm. 1; the necessary inequality $b\,(n+1) > 2$ always holds) when using $b^n\, n!\, e^{\pm 2/b}$ for $b^n\, n!\, e_n\left(\frac{\pm 2}{b}\right)$ is $\frac{2^{n+1}}{b(n+1)-2}$. But the enhanced form in Theorem 6's statement—using one additional series term—improves the bound to $B = \frac{b^n}{n+1} \sum_{j=1}^{\infty} \frac{1}{(n+2)^j} \left(\frac{2}{b}\right)^{n+1+j} \leq \frac{2^{n+2}}{b\,(n+1)\,(b\,(n+2)-2)}$.

Therefore $B + \frac{1}{2}$ bounds the difference between r and $K_n\left(\frac{a}{b}\right)$. Because any interval of length m contains exactly one integer divisible by m, the result follows from $B + \frac{1}{2} \leq m$, which in turn follows from the special case where $b = 1$. By replacing $\lceil n - \log_2(n+1) \rceil$ with $n - \log_2(n+1)$ in m, it is easy to see that, for $b = 1$ and any $n \geq 4$, we have $B + \frac{1}{2} \leq m$. Details: $B \leq m$ follows from $2^{n-\log_2(n+1)} \geq \frac{2^{n+2}}{(n+1)\,n} + \frac{1}{2}$, which follows from $2^{n-\log_2(n+1)} \geq \frac{2^{n+2}}{n^2}$, which is equivalent to $n - \log_2(n+1) \geq n + 2 - (2 \log_2 n)$, which simplifies to $2 \leq \log_2\left(\frac{n^2}{n+1}\right)$ or $4 \leq \frac{n^2}{n+1}$. This last is equivalent to $n \geq 4$. The inequality $B + \frac{1}{2} \leq m$ is easily checked when n is 3 and $b \geq 3$; it fails for $(b, n) = (1, 3)$. □

One can prove a version of Theorem 6 where the initial approximation is the simpler $r = \left\{b^n\, n!\, e^{\pm 2/b}\right\}$ by using Lemma 5's stronger congruence condition $K_n\left(\frac{\pm 2}{b}\right) \equiv 2^p \pmod{2^{p+1}}$, where $p = p_2(n)$ (note that $-2^p \equiv 2^p \pmod{2^{p+1}}$). But this requires the exact computation of p, which is avoided in Theorem 6. Aside: Using the alternating error for the -2 case gives a smaller B of $\frac{2^{n+2}}{b^2\,(n+1)\,(n+2)}$, but this has no essential impact.

Example. Suppose $a = 2$, $b = 1$, $n = 20$. Then $r = 17\,976\,849\,421\,618\,128\,596$ and $m = 2^{\lceil 20 - \log_2 21 \rceil} = 65\,536$. The rounded Maclaurin-polynomial-plus-one-term error bound is about 9986.94, well under m. The true value of this error is $b^n\, n!\, e^{a/b} - n!\, b^n\, e_{n+1}\left(\frac{a}{b}\right) \sim 9939.51$. The mod-$m$ residue of r is 9940; this agrees to the nearest integer with the true error. Indeed, this is the crux of the whole method: the rounded error *equals* the mod-m value of r. Now $r - \mathrm{mod}_m\, r = 17\,976\,849\,421\,618\,118\,656$ and so we conclude that the exact value of $e_{20}(2)$, the Maclaurin polynomial, is

$e_{20}(2) = \frac{17\,976\,849\,421\,618\,118\,656}{2\,432\,902\,008\,176\,640\,000}$, where the denominator is 20!. In lowest terms this is $e_{20}(2) = \frac{68\,576\,238\,333\,199}{9\,280\,784\,638\,125}$, which agrees with the sum of the 21 terms defining $e_{20}(2)$. And this gives $\Gamma(21, 2) = 17\,976\,849\,421\,618\,118\,656 \big/ e^2$. All this works quite quickly when n is large: $K_{100\,000}(2)$ is an integer with 456575 digits and the formula finds it in well under one second. For $K_{10^6}(2)$ the formula works in 30 seconds while the naive sum takes 21 minutes. All timings here are using *Mathematica* on an Apple iMac with a 4 GHz processor.

Here is a *Mathematica* implementation of the formula for $K_n\left(\frac{\pm 2}{b}\right)$, where $n \geq 4$ and b is odd. Note that $\mathrm{mod}\,(\cdot)$ works on real numbers and since $r - \mathrm{mod}_m\, r$ is divisible by m it follows that one can eliminate the Round operation from the formula, as well as the code that follows.

```
r - s Mod[r s, 2^⌈n-Log2[n+1]⌉] /. {r → n! b^n e^(a/b) - a^(n+1)/(b (n + 1)), s → Sign[a]^n}
```

The Case $|a| \geq 3$

For prime a-values beyond 2 the method that works so well when $|a| \leq 2$ fails because the truncation error is roughly a^n while the power of a in $n!$, the crux of the congruential method when $|a| \leq 2$, is only about $a^{n/2}$. We can still extend the method so that it improves classic algorithms, but the gain is less striking than in the smaller cases. The following method applies to any rational $\frac{a}{b}$, including composite a.

For simplicity we will use the Lagrange error bound, which is valid in all cases: $B = e^{|a|/b} \frac{|a|^{n+1}}{b(n+1)}$. We need a modulus $m \geq B$. If we try for one of the form $m = w!$, then the smallest w is easy to find and it is generally (but not always) less than n. For example, this method for $K_{100\,000}(3)$ has $w = 12\,969$ as the smallest possible modulus, well under n. But for $K_{900}(1000)$ the smallest w that works is 1187; being larger than n, it is useless. A good estimate of w—because $\ln(n!) > n \ln n - n$, this estimate is never less than the smallest w and so can be used directly in an algorithm—is $\lceil e^{1+W(\log(B)/e)} \rceil$, where W is the Lambert W-function (which, combined with Stirling's approximation $\ln(n!) \sim n \ln n - n$, can serve as an approximate inverse to the factorial). This approximation gives 12969.2 and 1187.15 for the two preceding examples. Some asymptotic analysis shows that $w \sim \frac{n}{\log_a n}$.

To finish, we need R, the mod-$w!$ residue of $K_n(a/b)$. With w and R in hand and using $s = \operatorname{sign}(a)^{n+1}$ and $r = \{n!\, b^n\, e^{a/b}\}$, we then have a single formula that applies to all cases: $K_n\left(\frac{a}{b}\right) = r - s \operatorname{mod}_{w!}(s\,(r - R))$. Because each of the initial $n - w + 1$ terms of the sum defining K are divisible by $w!$, computing R requires summing only the terminal w terms of the sum, and this is where the time gain arises. The sum computation for $K_{100{,}000}(3)$ requires 10 000 additions, so the reduction to 12 969 terms is a substantial savings.

We summarise the method in the following theorem.

Theorem 7. For any integers a, b, n with $b \geq 1$, $n \geq 0$, and a and b relatively prime, $K_n\left(\frac{a}{b}\right) = r - s \operatorname{mod}_{w!}(s\,(r - R))$, where $s = \operatorname{sign}(a)^{n+1}$, $a_1 = |a|$, $B_0 = a_1 / b + (n+1) \ln a_1 - \ln b - \ln(n+1)$, $r = \{n!\, b^n\, e^{a/b}\}$, $w = \lceil e^{1+W(B_0/e)} \rceil$, and $R = \operatorname{mod}_{w!}\left(n!\, b^n \sum_{k=n-w+1}^{n} \left(\frac{a}{b}\right)^k \frac{1}{k!}\right)$.

While some reduction in the modulus is possible by taking additional congruences into account, the gain is small. The simple algorithm just described requires about one second to compute $K_{100\,000}(3)$. Direct computation of the Maclaurin polynomial takes ten seconds.

Here is *Mathematica* code that works for all rationals (though when $|a| \leq 2$ the work of the earlier sections should be used instead). It is fairly simple and is faster than known algorithms.

- `K[a, b, n]` computes $K_n\left(\frac{a}{b}\right)$ by simple addition of $n+1$ terms.

- `KTailModFactorial[a, b, n, w]` computes $\operatorname{mod}_{w!}\left(K_n\left(\frac{a}{b}\right)\right)$ by addition of the relevant tail terms only.

- `KGeneral[a, b, n]` computes $K_n\left(\frac{a}{b}\right)$ (assuming $\gcd(a, b) = 1$) by using a factorial congruence to overcome the truncation error. If $a < 0$, the alternating series error is used; otherwise either the geometric series bound is used (when it converges) or the Lagrange bound. And the `Round` in the definition of r has been dropped, since the modular reduction of noninteger values of r yields the same thing.

```
K[a_, b_, n_] := K[Numerator[a / b], Denominator[a / b], n] /; GCD[a, b] ≠ 1;
K[a_, b_, n_] := Module[{sum, t}, sum = t = n ! b^n ;
  Do[sum += (t *= a / (b k)), {k, n}]; sum];
KTailModFactorial[a_, b_, n_, w_] := Module[{n0, t, ρ = 0}, (n0 = n - w + 1;
   t = PowerMod[a, n0, w !] n ! b^(w-1) / n0 !;
   Do[ρ += t; t *= a / (b (k + 1)), {k, n0, n}]; Mod[ρ, w !])];
KGeneral[a_, b_, n_] := Module[{a1 = Abs[a], s = Sign[a]^(n+1), L, logB, w, r},
  L = (n + 1) Log[a1];
  logB = Which[a < 0, L - Log[b (n + 1)],
            a1 ≥ b (n + 1), a / b + L - Log[b (n + 1)],
            True, L - Log[b (1 + n) - a1]];
  If[LogGamma[n] < logB, K[a, b, n],
   w = ⌈ⅇ^(1+ProductLog[logB/ⅇ])⌉; s = Sign[a]^(n+1); r = b^n n ! ⅇ^(a/b);
   r - s Mod[s (r - KTailModFactorial[a, b, n, w]), w !]]] /; GCD[a, b] == 1

KGeneral[3, 1, 100 000] // Short // Timing

{1.12787, 56 726 164 053 140 014 101 900 918 <<456 523>> 14 370 403 877 621 612 232 600 001}
```

For $K_{10^6}(3)$ the preceding code takes 90 seconds to find the 5.5-million-digit integer; the modulus defining the number of terms in the tail sum is $w = 104\,102$.

There are a number of tricks that can be used to improve the algorithm. One can reduce the truncation error interval by adding more terms or considering a lower bound on the error as well, and one can consider additional congruences (such as, when a is prime, the one from Lemma 5); but these ideas improve things only a little. So the question remains whether there are formulas or algorithms that would compute the symbolic value of $K_n(z)$ for any rational z more quickly than the modular method of Theorem 7.

Acknowledgements. The author thanks Jonathan Borwein and Simon Plouffe for some valuable suggestions, and Robert Israel for finding the key idea in the proof of Lemma 4 in the general case.

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