# PERIODIC OMNIHEDRAL BILLIARDS IN REGULAR POLYHEDRA AND POLYTOPES

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**Abstract**

We explore questions concerning the existence of periodic billiards in regular polytopes. Specifically, we address whether such billiards exist which strike each face once per period (such a billiard is called "omnihedral.") Examples of omnihedral billiards whose segment lengths are all equal are known for the cube and the regular tetrahedron. To complete the list of omnihedral billiards for regular polyhedra, we exhibit examples of such billiards, with unequal segment lengths, for the other three regular polyhedra in three–dimensional Euclidean space. Also, we generalize the constructions for the cube and tetrahedron to billiards in higher dimensions, as well as exhibit an example of an omnihedral billiard for the four–dimensional cross polytope.

## 1 INTRODUCTION AND BACKGROUND

We consider the problem of locating periodic billiards in a polytope in $\mathbf{R}^n$, $n \geq 3$, which strike the faces in some prescribed order. In particular, we desire to know if there is a periodic billiard in each of the regular polyhedra which strikes each face precisely once per period. Such a billiard will be called **omnihedral**. An example of such a billiard for the cube was sought after in a problem by Lewis Carroll and constructed by Hugo Steinhaus. An example for the tetrahedron was constructed by J. H. Conway and Roger Hayward in 1962. Figure 1 depicts these billiards in the cube and tetrahedron which are generalized to higher dimensions in section 3. We demonstrate examples for the remaining regular convex polyhedra in three–dimensional Euclidean space in section 2 and for higher dimensional polytopes in section 3.

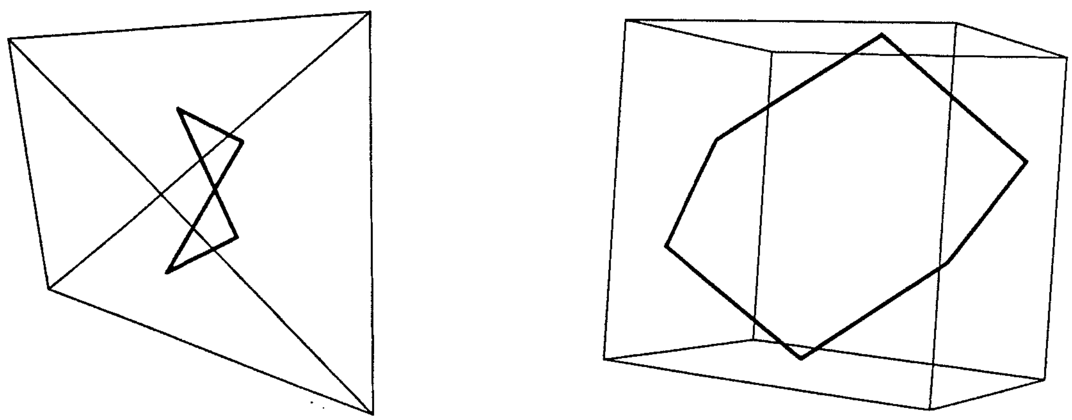

Figure 1: Omnihedral billiards in the cube and tetrahedron

In the case of the cube and the tetrahedron, omnihedral periodic billiards are known whose segment lengths are all equal. We call such billiards **equilateral**. To date, it is not known whether equilateral omnihedral periodic billiards of the octahedron, dodecahedron or icosahedron once per period exist, although we conjecture here that such billiards do not exist.

In the case of the omnihedral billiards in the octahedron, dodecahedron, and icosahedron, we can use analytic means to determine the proper starting direction of the desired billiards, given a sequence of faces we wish to visit. However, finding the desired sequences of faces is less straightforward. The sequences of faces presented here were found probabilistically. For this work, a computer was used to randomly select pairs of points lying on the surface of the polyhedron. The points were chosen among a uniform distribution on the polyhedral surface. For each pair, the computer determined the order in which the billiard containing the pair would strike the faces. Any such ordering which struck each face once was then analyzed to determine the proper direction in which to commence the billiard path. Given this direction, further geometric and algebraic analysis could be done to determine the proper starting point, if a unique point exists, as in the case of the icosahedron. For the octahedron and dodecahedron, it turns out that any point on a small area on the starting face is suitable for a starting point of a periodic omnihedral billiard.

## 1.1 Preliminary Computations

Given a polytope $P$ with the origin in its interior, its faces may be described as hyperplanes $F_i = \{x : x \cdot n_i = N\}$, where $n_i$ is a normal vector to the $i$th face, and $N$ is a scaling constant. Now, suppose we have a periodic billiard path which strikes the faces $F_0, F_1, F_2, \dots, F_k = F_0$ in order. Suppose further that the billiard path starts at some point $p_0 \in F_0$ with direction $v_0$. Let $v_i$ be the direction the path takes from $F_i$ to $F_{i+1}$. We then note that $v_{i+1}$ is the reflection of $v_i$ in the plane $G_{i+1} = \{x : x \cdot n_{i+1} = 0\}$. This reflection is a linear map $M_{i+1} : \mathbf{R}^3 \to \mathbf{R}^3$ given by $M_{i+1}v = v - 2n_{i+1}(n_{i+1} \cdot v)/(n_{i+1} \cdot n_{i+1})$. It then follows that $v_0 = v_k = M_k M_{k-1} M_{k-2} \cdots M_1 v_0$. Therefore, our starting direction must be an eigenvector of $\Pi = M_k M_{k-1} \cdots M_1$ with eigenvalue 1.

We note that the matrix $\Pi$ has determinant $(-1)^k$ since each $M_i$ has determinant $-1$. Since we are confining our attention to $\mathbf{R}^3$, if $k$ is even, then $\Pi$ will be a $3 \times 3$ orthogonal matrix with determinant 1, and so $\Pi \in SO_3$. Therefore, either $\Pi$ is the identity matrix or $\Pi$ is a rotation about an axis $w$ through the origin. If $\Pi$ is the identity matrix, then every direction is preserved and so there appears to be no information obtained from $\Pi$. On the other hand, if we "fold out" the polytope by successive reflections in all of the faces, then we obtain a "chain" of copies of the polytope. An example of such a chain of forty icosahedra is shown in figure 2 and the view down such a chain is shown in figure 3.

If $\Pi$ is the identity matrix, then the last copy of the polytope is a translation of the original polytope, and so our desired direction is the direction of this translation. To compute this, we reflect the origin successivley in the faces $F_1, F_2, \dots, F_k$. We let $q_0$ be the origin and $q_{i+1}$ be the reflection of $q_i$ in $F_{i+1}$. We then let $v_0$ be the direction of $q_k$. If $\Pi$ is a rotation about an axis $w$, then our desired direction $v_0$ will either be $w$ or $-w$, depending on

Figure 2: A "chain" of icosahedra

the orientation of the faces $F_0$ and $F_1$.

Given $v_0$, we may then search for suitable starting points $p_0$ on $F_0$ such that billiard paths starting at $p_0$ in the direction $v_0$ hit the faces $F_0, F_1, F_2, \ldots, F_k = F_0$ in order, returning to $p_0$. We will see the identity matrix process in action in the cases of the octahedron and dodecahedron, while for the icosahedron, we will use the rotation matrix process to determine $v_0$. In each case, however, there will be only one choice for $v_0$.

Naturally, given a sequence of vertices $v_0, v_1, \ldots, v_k = v_0$, we would like to verify that such a sequence is in fact a billiard striking faces $F_0, F_1, \ldots, F_k = F_0$. We assume that our convex polytope is the set

$$\{x : n_i \cdot x \leq 1 \text{ for each } n_i \text{ normal to the face } F_i.\}$$

Then our procedure is to verify that each $v_i$ lies in the relative interior of $F_i$ by checking $n_i \cdot v_i = 1$, $n_j \cdot v_j < 1$ for $j \neq i$. Since our polytope is assumed to be convex, the path will lie entirely inside the polytope if and only if these equations and inequalities are satisfied.

Finally, for each $v_i$, we determine the direction $d_i$ of the bisector of the angle formed by $v_{i-1}$, $v_i$, and $v_{i+1}$, all subscripts modulo $k$. If this bisector is parallel to $n_i$, then the path a billiard through $v_i$. We can compute $d_i$ by the formula

$$d_i = \frac{v_i - v_{i-1}}{\|v_i - v_{i-1}\|} + \frac{v_i - v_{i+1}}{\|v_i - v_{i+1}\|}.$$

Therefore, if $d_i$ is a multiple of $n_i$ for all $i$, our path is in fact a billiard.

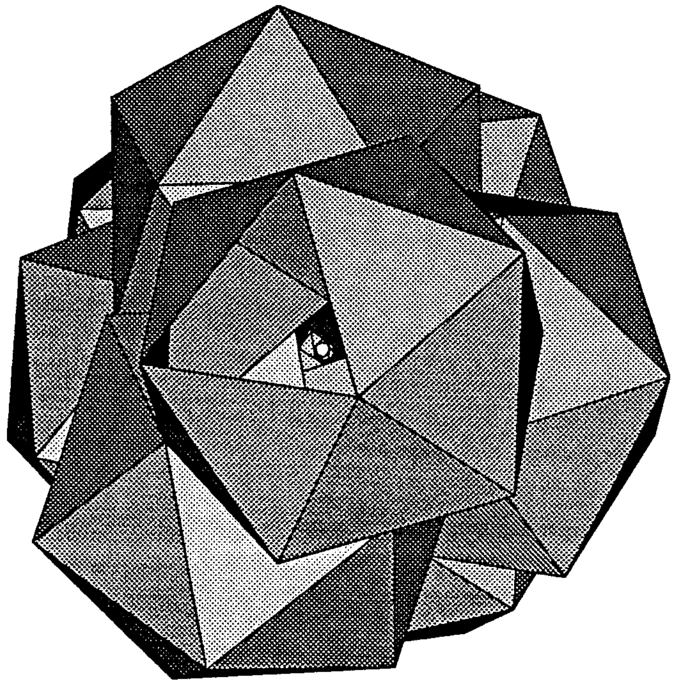

Figure 3: The view through the chain of icosahedra

# 2 OMNIHEDRAL PERIODIC BILLIARDS IN THE REGULAR POLYHEDRA

In this section, we demonstrate the existence of omnihedral periodic billiards in all five of the classical regular polyhedra. The data for vertices and faces of polyhedra in this section are from Coxeter [3]. That the tetrahedron and the cube admit omnihedral periodic billiards has been known since 1962. In fact, examples of such billiards which are equilateral have been constructed. Generalizations of such billiards to all dimensions $d \geq 2$ are given in section 3. Here, we complete the list by exhibiting suitable billiards in the octahedron, dodecahedron, and icosahedron.

## 2.1 The Octahedron

We examine the regular octahedron whose vertices are the endpoints of the standard basis vectors and their negatives in $\mathbf{R}^3$. Therefore, the faces of this octahedron lie on the planes $\pm x + \pm y + \pm z = 1$. We exhibit an omnihedral periodic billiard which strikes the faces in the order $x + y + z = 1$, $-x + y - z = 1$, $x - y - z = 1$, $-x - y + z = 1$, $x + y - z = 1$, $-x + y + z = 1$, $x - y + z = 1$, and $-x - y - z = 1$, respectively. We note that when the reflection matrices corresponding to these planes are multiplied, the result is the identity matrix, and so it suffices to track where the origin is taken when reflected in these faces. The origin finally maps to $(272/81, 512/81, 128/81)$ which means this is the translation direction. Therefore, if the origin lies on the billiard then it hits the face $x + y + z = 1$ at $(17/57, 32/57, 8/57)$. Following the billiard around the octahedron, we find that its remaining points are $(-1/9, 4/9, -4/9)$, $(1/9, -4/9, -4/9)$, $(-17/57, -32/57, 8/57)$, $(17/57, 32/57, -8/57)$, $(-1/9, 4/9, 4/9)$, $(1/9, -4/9, 4/9)$, and $(-17/57, -32/57, -8/57)$. To see that these vertices form a periodic omnihedral billiard, we first note that these points are on the faces in the prescribed order.

All that remains is to determine whether this path is a billiard. To do this, we must determine the vector which bisects the angle formed by each adjacent pair of segments in the path. If $v_0, v_1$, and $v_2$ are successive vertices of the path, then the direction of the bisecting vector is given by

$$\frac{(v_1 - v_0)}{\|v_1 - v_0\|} + \frac{(v_1 - v_2)}{\|v_1 - v_2\|}.$$

Applying this to $v_0 = (17/57, 32/57, 8/57)$, $v_1 = (-1/9, 4/9, -4/9)$, and $v_2 = (1/9, -4/9, -4/9)$, we have that the bisection direction is $\frac{10}{3\sqrt{17}}(-1, 1, -1)$ which is a multiple of the normal to the plane $-x + y - z = 1$ on which $v_1$ lies. Similar computations are easy to verify for the other faces, showing that our path is in fact a billiard.

We make the final note that any parallel path which is sufficiently close to the exhibited path is also an omnihedral periodic billiard, so the octahedron admits an uncountable number of such paths.

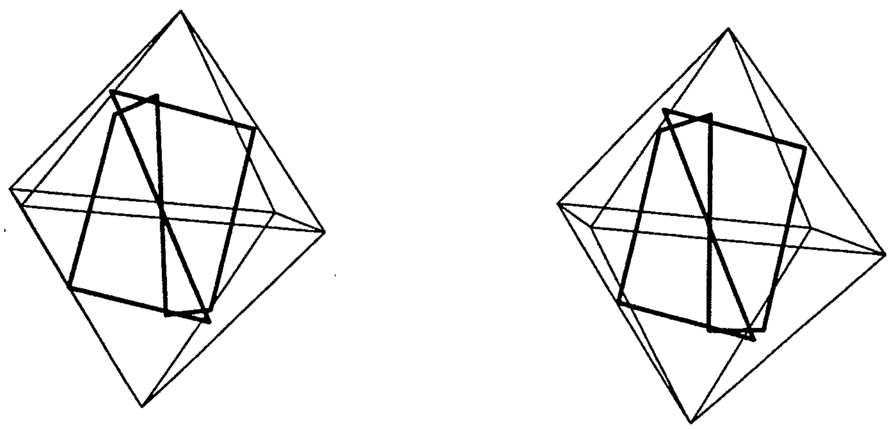

Figure 4: An omnihedral billiard in the regular octahedron, stereoscopic view

## 2.2 The Dodecahedron

We use the standard notation $\phi = (1 + \sqrt{5})/2$ in the sequel. The following list of vectors are normals to the faces of the dodecahedron as they are encountered along a periodic billiard path:

| Face | Normal Vector | Face | Normal Vector |
|---|---|---|---|
| 1 | $(1, 0, \phi)$ | 7 | $(-\phi, -1, 0)$ |
| 2 | $(1, 0, -\phi)$ | 8 | $(0, \phi, 1)$ |
| 3 | $(-\phi, 1, 0)$ | 9 | $(0, -\phi, -1)$ |
| 4 | $(0, -\phi, 1)$ | 10 | $(\phi, 1, 0)$ |
| 5 | $(0, \phi, -1)$ | 11 | $(-1, 0, \phi)$ |
| 6 | $(\phi, -1, 0)$ | 12 | $(-1, 0, -\phi)$ |

If the face normal to each of these vectors $v$ lies in the plane $v.x = 1160$, then the locations of the vertices of the billiard path are at the following vectors:

| Vertex | Location | | |
|---|---|---|---|
| 1 | $(580,$ | $145,$ | $-580+580\phi)$ |
| 2 | $(3132-1624\phi,$ | $145,$ | $-3596+1972\phi)$ |
| 3 | $(1015-1015\phi,$ | $145,$ | $1247-754\phi)$ |
| 4 | $(1173-906\phi,$ | $-171-218\phi,$ | $942-389\phi)$ |
| 5 | $(1093-506\phi,$ | $-939+838\phi,$ | $-322-101\phi)$ |
| 6 | $(1479-551\phi,$ | $-1711+928\phi,$ | $-2001+1102\phi)$ |
| 7 | $(-1479+551\phi,$ | $-1711+928\phi,$ | $5017-2958\phi)$ |
| 8 | $(-669+242\phi,$ | $-91+310\phi,$ | $850-219\phi)$ |
| 9 | $(-749+642\phi,$ | $677-746\phi,$ | $-414+69\phi)$ |
| 10 | $(-1015+1015\phi,$ | $145,$ | $1769-1102\phi)$ |
| 11 | $(-116-232\phi,$ | $145,$ | $-1276+1044\phi)$ |
| 12 | $(-1164+368\phi,$ | $145,$ | $-372+4\phi)$ |

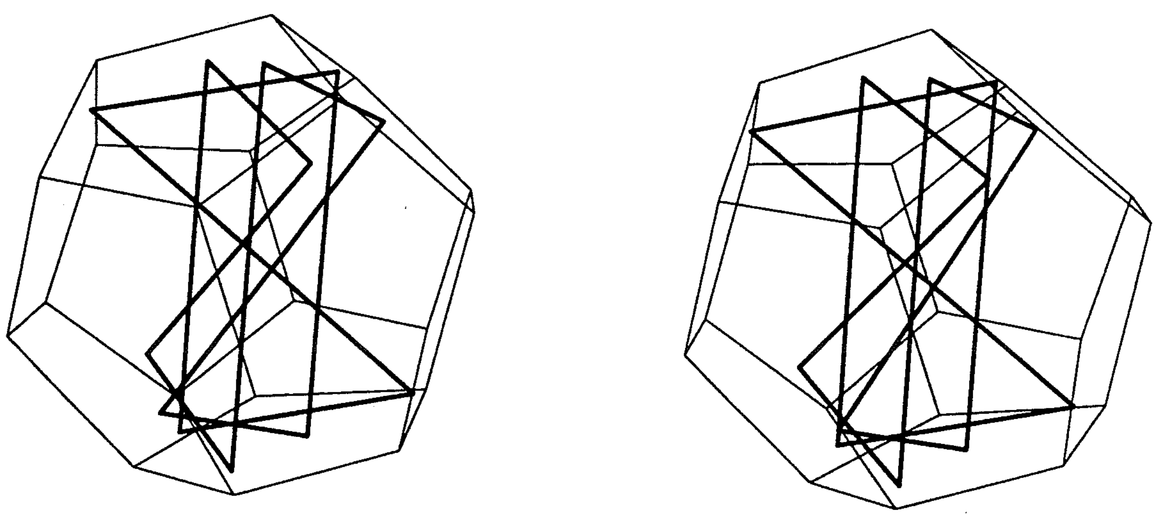

Figure 5: An omnihedral billiard in the regular dodecahedron, stereoscopic view

Figure 6 shows the path as seen from near one of the axes of rotational symmetry. This figure is included to display the three-fold rotational symmetry which the path possesses.

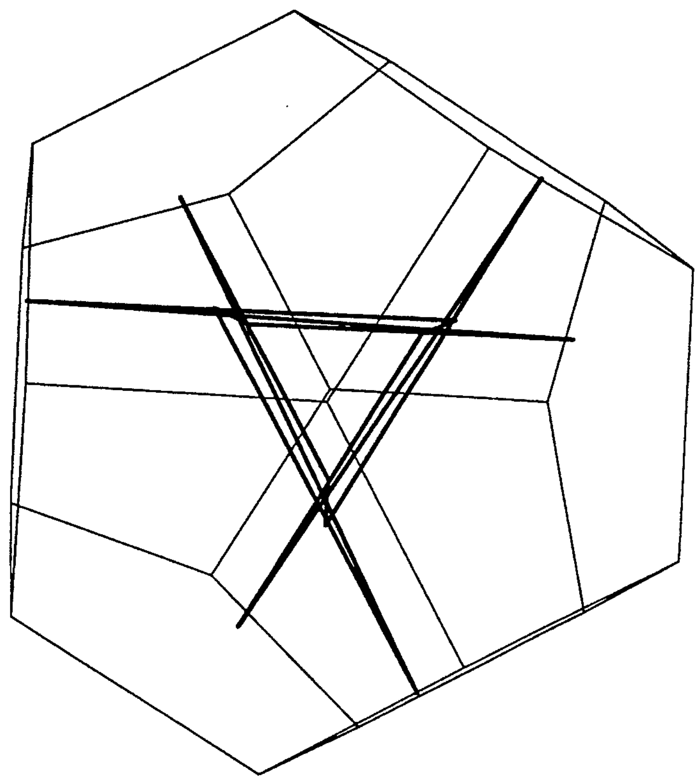

Figure 6: An omnihedral billiard in the regular dodecahedron, polar view

The computations for the dodecahedron are similar to those for the octahedron since the reflection matrices have the identity as their product in the prescribed order.

## 2.3 The Icosahedron

Here, as always, $\phi = (1+\sqrt{5})/2$. The following list of vectors are normals to the faces of the icosahedron as they are encountered along a periodic billiard path:

| Face | Normal Vector | Face | Normal Vector |
|---|---|---|---|
| 1 | $(-2+\phi,-2+\phi,-2+\phi)$ | 11 | $(-3+2\phi,-1+\phi,0)$ |
| 2 | $(2-\phi,2-\phi,-2+\phi)$ | 12 | $(0,3-2\phi,-1+\phi)$ |
| 3 | $(-2+\phi,2-\phi,2-\phi)$ | 13 | $(0,3-2\phi,1-\phi)$ |
| 4 | $(2-\phi,-2+\phi,2-\phi)$ | 14 | $(3-2\phi,-1+\phi,0)$ |
| 5 | $(0,-3+2\phi,1-\phi)$ | 15 | $(-2+\phi,-2+\phi,2-\phi)$ |
| 6 | $(1-\phi,0,-3+2\phi)$ | 16 | $(-1+\phi,0,-3+2\phi)$ |
| 7 | $(-3+2\phi,1-\phi,0)$ | 17 | $(-2+\phi,2-\phi,-2+\phi)$ |
| 8 | $(2-\phi,2-\phi,2-\phi)$ | 18 | $(3-2\phi,1-\phi,0)$ |
| 9 | $(1-\phi,0,3-2\phi)$ | 19 | $(-1+\phi,0,3-2\phi)$ |
| 10 | $(2-\phi,-2+\phi,-2+\phi)$ | 20 | $(0,-3+2\phi,-1+\phi)$ |

If the face normal to each of these vectors $v$ lies in the plane $v.x = 204479$, then the vertices of the billiard path are at the following vectors:

| Vertex | Location | | |
|---|---|---|---|
| 1 | $(-32761+98944\phi,$ | $37456+12456\phi,$ | $-62456+229479\phi)$ |
| 2 | $(-30159-93664\phi,$ | $-168576+55744\phi,$ | $-5744-166559\phi)$ |
| 3 | $(38005+11110\phi,$ | $96404+106304\phi,$ | $-70070-87065\phi)$ |
| 4 | $(-38005-11110\phi,$ | $96404+106304\phi,$ | $70070+87065\phi)$ |
| 5 | $(30159+93664\phi,$ | $-168576+55744\phi,$ | $5744+166559\phi)$ |
| 6 | $(32761-98944\phi,$ | $37456+12456\phi,$ | $62456-229479\phi)$ |
| 7 | $(-78287-150722\phi,$ | $-25652-60764\phi,$ | $-24530+29227\phi)$ |
| 8 | $(-18117+44418\phi,$ | $-80652-141944\phi,$ | $34430+67177\phi)$ |
| 9 | $(135815+43200\phi,$ | $-93200+100376\phi,$ | $161864+60903\phi)$ |
| 10 | $(-67615-106120\phi,$ | $156120-105656\phi,$ | $-30744-129103\phi)$ |
| 11 | $(63657+71874\phi,$ | $-35640-19888\phi,$ | $-105182-112717\phi)$ |
| 12 | $(42647+130834\phi,$ | $45540+116292\phi,$ | $8778+6413\phi)$ |
| 13 | $(123359-6712\phi,$ | $68200-62920\phi,$ | $199320+73359\phi)$ |
| 14 | $(-123359+6712\phi,$ | $68200-62920\phi,$ | $-199320-73359\phi)$ |
| 15 | $(-42647-130834\phi,$ | $45540+116292\phi,$ | $-8778-6413\phi)$ |
| 16 | $(-63657-71874\phi,$ | $-35640-19888\phi,$ | $105182+112717\phi)$ |
| 17 | $(67615+106120\phi,$ | $156120-105656\phi,$ | $30744+129103\phi)$ |
| 18 | $(-135815-43200\phi,$ | $-93200+100376\phi,$ | $-161864-60903\phi)$ |
| 19 | $(18117-44418\phi,$ | $-80652-141944\phi,$ | $-34430-67177\phi)$ |
| 20 | $(78287+150722\phi,$ | $-25652-60764\phi,$ | $24530-29227\phi)$ |

The product of the reflection matrices is not the identity matrix in this case, and so we do not obtain our direction by means of a translation. Rather, we use the eigenvector having eigenvalue one to determine our desired direction.

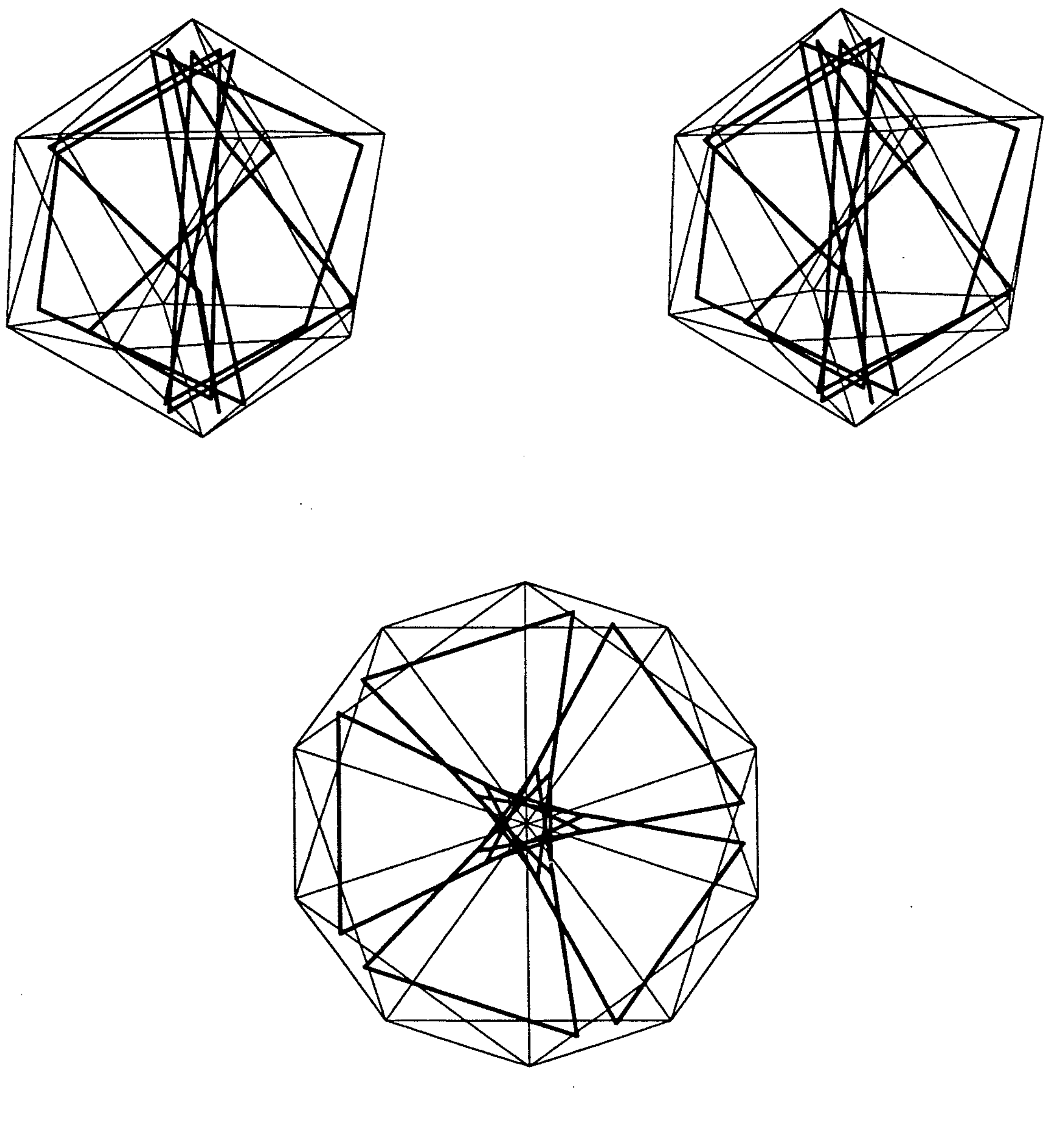

Figure 7: An omnihedral billiard in the regular icosahedron, stereoscopic and polar views

# 3 OMNIHEDRAL PERIODIC BILLIARDS IN HIGHER DIMENSIONAL POLYTOPES

In this section, we generalize the omnihedral periodic billiards in the tetrahedron and cube to their higher dimensional counterparts. We close by exhibiting an omnihedral periodic billiard in the four–dimensional cross–polytope.

## 3.1 Billiards in Hypercubes

Let $Q_n = [0,1]^n$ be the $n$–dimensional hypercube and let $S$ be the oscillating bi–infinite sequence

$$\ldots, 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-2}{n}, \frac{n-1}{n}, 1, \frac{n-1}{n}, \frac{n-2}{n}, \ldots, \frac{2}{n}, \frac{1}{n}, 0, \ldots$$

where $S(0) = 0$. For instance, if $n = 3$, then the sequence of values of $S(k)$ are

$$\ldots, 0, \frac{1}{3}, \frac{2}{3}, 1, \frac{2}{3}, \frac{1}{3}, 0, \ldots.$$

**Theorem 1** *The sequence of points $P_k = (S(1+k), S(2+k), \ldots, S(n+k))$, $k \in \mathbf{Z}$, represents a periodic omnihedral billiard in $Q_n$ which is also equilateral.*

Since $S(k)$ is periodic with period $2n$, so is the sequence $P_k$, so we may concentrate on the values $0 \le k \le 2n-1$. First, we note that $P_k \in Q_n$ since no coordinate value exceeds one or is less than zero. Next, we note that either precisely one of the coordinates of $P_k$ is a one or a zero, so each $P_k$ lies on the relative interior of an $(n-1)$–dimensional face of $Q_n$. Also, since each $P_k$, $0 \le k \le n-1$, lies on $x_{n-k} = 1$ and each $P_k$, $n \le k \le 2n-1$, lies on $x_{2n-k} = 0$, each of the $2n$ faces is hit once per period. Therefore, the cycle is omnihedral. We further note that the distance between $P_k$ and $P_{k+1}$ is $\sqrt{1/n}$ since corresponding coordinates of $P_k$ and $P_{k+1}$ are separated by $1/n$. Therefore, the cycle is equilateral.

All that remains is to show that the cycle is a billiard. We compute a vector which bisects the angle formed by $P_k - P_{k-1}$ and $P_k - P_{k+1}$. We want to show that this vector is normal to the face containing $P_k$. Since the cycle is equilateral, these vectors are the same length, and so their sum will bisect the desired angle. Therefore, we compute the $i$th coordinate of $R_k = 2P_k - P_{k-1} - P_{k+1}$; the said coordinate has value $2S(k+i) - S(k+i-1) - S(k+i+1)$. If $S(k+i) = 0$, then the $i$th coordinate is $-2/n$. If $S(k+i) = 1$, then the $i$th coordinate is $2 - (2/n)$. Otherwise, $S(k+i)$ is the average of $S(k+i-1)$ and $S(k+i+1)$ and the $i$th coordinate of $R_k$ is zero. Therefore, the only non-zero coordinate of $R_k$ will be the $i$th coordinate, where $k + i$ is a multiple of $n$. Therefore, for $0 \le k \le n-1$, $i = n-k$, and so $R_k$ is normal to $x_{n-k} = 1$ and for $n \le k \le 2n-1$, $i = 2n-k$, and so $R_k$ is normal to $x_{2n-k} = 0$. In either case, the end result is that $R_k$ is normal to the face containing $P_k$, and so the cycle is indeed a billiard, proving the result. □

## 3.2 Billiards in Regular Simplices

In considering the $n$-dimensional regular simplex, we will proceed in reverse, starting by constructing a polygonal cycle connecting $n+1$ points and then constructing a regular simplex in which the cycle is an periodic omnihedral equilateral billiard.

Let $f(k) = n + 2 - 6k(n+1-k)/n$ where $0 \le k \le n$, We note that $f(n+1-k) = f(k)$ by the symmetry of the roles of $k$ and $n+1-k$. We also note that for $1 \le k \le n$, $f(0) > f(k)$. This is clear since $f(k) = n + 2 - 6k(n+1-k)/n < n+2 = f(0)$.

We then let $p_0 = (f(0), f(1), \ldots, f(n))$, $p_1 = (f(1), f(2), \ldots, f(n), f(0))$, and in general, $p_j = p_0$ rotated $j$ places to the left. Since each point is generated from the previous point by cycling one position to the left, a routine calculation shows that the distance between $p_k$ and $p_{k+1}$ for each $k$ from 0 to $n-1$ is the same as the distance between $p_n$ and $p_0$, namely

$$\sqrt{(f(0) - f(n))^2 + \sum_{i=0}^{n-1} (f(i+1) - f(i))^2}$$

and so the cycle $P = p_0, p_1, \ldots, p_n$ is equilateral.

Let $v_0 = (-n, 1, 1, \ldots, 1)$ be in Euclidean $(n+1)$-dimensional space and $v_i$ be $v_0$ rotated $i$ places to the left, so $v_1 = (1, 1, \ldots, 1, -n)$, and so on.

Now, let $t_i = n(n+2)v_i$ and $U(i)$ the hyperplane given by $x \cdot v_i = -(n+1)(n+2)$. First, we note that $U(i)$ contains every $t_j$ except $t_i$. Next, we note that all $t_i$ lie on the hyperplane $Z$ given by $x.(1,1,1,\ldots,1) = 0$. Finally, we note that the $t_i$ are equidistant from the other $T_i$, and so the $t_i$ form a regular $n$–simplex $T$.

We now show that $p_0.(1,1,1,\ldots,1) = 0$ and so, by symmetry, every $p_i$ lies on the hyperplane $Z$:

$$\begin{aligned}
p_0.(1,1,\ldots,1) &= \textstyle\sum_{k=0}^{n} f(k) \\
&= \textstyle\sum_{k=0}^{n}(n+2-6k+6k(k-1)/n) \\
&= n+1(n+2)-3n(n+1)+(n+1)(2n+1)-3(n+1) \\
&= n^2+3n+2-3n^2-3n+2n^2+3n+1-3n-3 \\
&= 0
\end{aligned}$$

as desired. By linearity, then, the entire path $P$ lies on $Z$.

Next, we show that $p_0$ lies on $U(0)$ and, by symmetry, $p_i$ lies on $U(i)$:

$$\begin{aligned}
p_0.v_0 &= (-nS(0)+\textstyle\sum_{k=1}^{n} S(k) \\
&= -n(n+2)+\textstyle\sum_{k=1}^{n}(n+2-6k+6k(k-1)/n) \\
&= -n^2-2n+(n(n+2)-3n(n+1)+1/n\textstyle\sum_{i=1}^{n}(6k^2-6k)) \\
&= -n^2-2n+(n^2+2n-3n(n+1)+(n+1)(2n+1)-3(n+1)) \\
&= -n^2-3n-2 \\
&= -(n+1)(n+2)
\end{aligned}$$

as desired.

We need to show that $P$ lies in the simplex $T$, touching its boundary only on the $U(i)$. It will suffice to show that the quantity $p_i \cdot v_j$ reaches a minimum precisely when $i = j$. Holding $i$ fixed, we see that

$$p_i \cdot v_j = -nS(j-i) + \sum_{k \neq (i-j)} f(k)$$

Therefore,

$$p_i \cdot v_i - p_i \cdot v_j = (n-1)(f(i-j) - f(0))$$

and, since $f(0) > f(k)$ for all $k \neq 0$, this latter difference is strictly negative unless $i = j$. Therefore, the entire path $P$ lies in the intersection of $Z$ and the half–spaces $x \cdot v_i \geq -(n+1)(n+2)$.

Since the cycle $P$ is equilateral, we can compute normals to bounce faces by the formula $r_k = 2p_k - p_{k-1} - p_{k+1}$ where the subscripts are modulo $n+1$.

We note that $r_k$ will be $r_0$ cycled $k$ positions to the left, so it suffices to compute $r_0 = (\xi_0, \xi_1, \ldots, \xi_n)$. If $1 \le j \le n-1$, then

$$\begin{aligned} \xi_j &= 2f(j) - f(j-1) - f(j+1) \\ &= 2(n+2-6j+6j(j-1)/n) - (n+2-6(j-1)+6(j-1)(j-2)/n) \\ &\quad -(n+2-6(j+1)+6(j+1)j/n \\ &= -12/n. \end{aligned}$$

For $j = 0$, we have

$$\begin{aligned} \xi_0 &= 2f(0) - f(1) - f(n) \\ &= 2f(0) - 2f(1) \\ &= 2(n+2) - 2(n+2-6) \\ &= 12. \end{aligned}$$

For $j = n$, we have

$$\begin{aligned} \xi_n &= 2f(n) - f(n-1) - f(0) \\ &= fS(1) - f(2) - f(0) \\ &= \xi_1 \\ &= -12/n. \end{aligned}$$

Therefore, $r_0 = 12(1, -1/n, -1/n, -1/n, \ldots, -1/n)$ which is normal to the hyperplane $U(0)$. From symmetry, it follows that each $r_i$ is normal to the hyperplane $U(i)$ and that $P$ is in fact a billiard inside $T$. Therefore, every regular $n$–simplex admits an equilateral omnihedral periodic billiard.

## 3.3 The Four–Cross Polytope

The following list of vectors are normals to the faces of the 4–cross polytope as they are encountered along an omnihedral periodic billiard:

| Face | Normal Vector | | | | Face | Normal Vector | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 9 | −1 | −1 | −1 | −1 |
| 2 | 1 | −1 | −1 | −1 | 10 | −1 | 1 | 1 | 1 |
| 3 | −1 | 1 | −1 | 1 | 11 | 1 | −1 | 1 | −1 |
| 4 | 1 | 1 | 1 | −1 | 12 | −1 | −1 | −1 | 1 |
| 5 | −1 | 1 | 1 | −1 | 13 | 1 | −1 | −1 | 1 |
| 6 | −1 | −1 | 1 | 1 | 14 | 1 | 1 | −1 | −1 |
| 7 | 1 | 1 | −1 | 1 | 15 | −1 | −1 | 1 | −1 |
| 8 | −1 | 1 | −1 | −1 | 16 | 1 | −1 | 1 | 1 |

If the face normal to each of these vectors $v$ lies in the plane $v.x = 20$, then the locations of the vertices of the billiard are at the following vectors:

| Vertex | Location | | | |
|---|---|---|---|---|
| 1 | 2 | 2 | 4 | 12 |
| 2 | −1 | 5 | −2 | 12 |
| 3 | 8 | −4 | −2 | −6 |
| 4 | 2 | 2 | 10 | −6 |
| 5 | −2 | 2 | 12 | −4 |
| 6 | −5 | −1 | 12 | 2 |
| 7 | 4 | 8 | −6 | 2 |
| 8 | −2 | 2 | −6 | −10 |
| 9 | −2 | −2 | −4 | −12 |
| 10 | 1 | −5 | 2 | −12 |
| 11 | −8 | 4 | 2 | 6 |
| 12 | −2 | −2 | −10 | 6 |
| 13 | 2 | −2 | −12 | 4 |
| 14 | 5 | 1 | −12 | −2 |
| 15 | −4 | −8 | 6 | −2 |
| 16 | 2 | −2 | 6 | 10 |

The reflection matrices have the identity as their product, so nearby paths that are parallel to this one are also omnihedral periodic billiards. Therefore, this polytope also has an uncountable number of such billiards.

# 4 OPEN QUESTIONS

As of yet, no omnihedral periodic billiards have been found in the remaining regular convex polytopes: These are the 24–cell (Schläfli symbol $\{3, 4, 3\}$), the 120–cell (Schläfli symbol $\{5, 3, 3\}$), the 600–cell (Schläfli symbol $\{3, 3, 5\}$), or in higher dimensional cross–polytopes. [?] Naturally, those polytopes with a smaller number of facets lend themselves more easily to determining whether such billiards exist within such polytopes. On the other hand, the billiards which were discovered in the dodecahedron and icosahedron exhibit some symmetries. Therefore, it makes sense to try to exploit symmetries as much as possible in trying to construct omnihedral periodic billiards in these unresolved cases.

Also, it would be interesting to extend this work to other non–regular polytopes, such as the Archimedian polyhedra. It would also be interesting to investigate non–convex polytopes, such as stellations of the regular polyhedra in order to see if any symmetrical periodic billiards exist in these bodies.

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