Hex Graphs

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A recent article [D] in this journal investigated graphs arising from chess-like moves along hexagonal cells. Here we will expand on that discussion. There is an actual game of hex chess (developed by Gliński in 1936 [WI]) and it seems reasonable to name pieces as in that game. True hex chess has the cells arranged in a large hexagon, which is of course similar to the subdivision of a square that governs traditional chess. But the triangular array is geometrically attractive so, as in [D], we will focus on that.

Two graphs of interest are the graph of the hex rook (called a queen in [D]) and also the king of [D], which is not a true hex king but is a triangular grid graph, TG_n. That graph is just the dual of the hex board (Fig. 1): vertices are cells and edges correspond to adjacent cells. We will also look at the bishops of [D] because — surprise — they turn out to be isomorphic to traditional chess bishops.

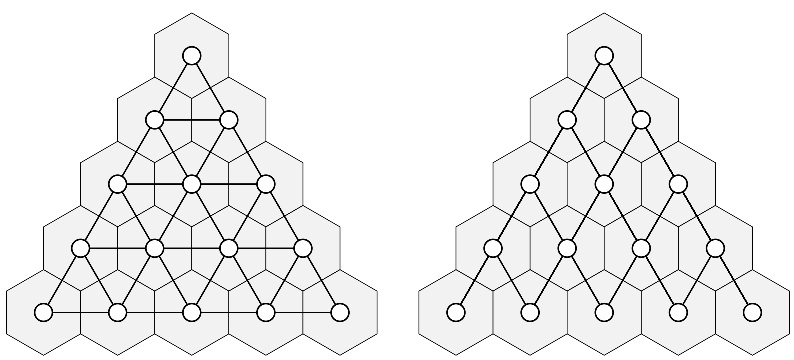

Figure 1. The triangular grid graph (left) and a type of hex bishop graph.

All of these provide interesting areas of exploration: sometimes there are patterns that lead to interesting formulas. Sometimes the conjectured formulas can be proved. And sometimes there are no formulas, but getting values is a significant computational challenge. We shall mention some of the highlights here.

I am grateful to Joseph DeMaio, Rob Pratt, Dan Ullman, and Doug West for helpful discussions.

Notation and terminology

The vertex count of a graph is V; the number of edges is E. The independence number is α, the chromatic number is χ, the clique covering number (χ of the complementary graph) is θ, and the domination number is γ. We use Δ for the maximum vertex degree. Then χ' is the edge chromatic number and it equals either Δ or $\Delta + 1$; a graph is called class 1 in the former case, class 2 in the latter.

A graph is *Hamiltonian* if it admits a Hamiltonian cycle: a cycle that passes through all vertices and never repeats an edge. A *Hamiltonian path* (HP) is a path that visits all vertices and never repeats an edge. A graph (necessarily regular) has a Hamitonian decomposition (HD) if the edges can be partitioned into disjoint Hamiltonian cycles (if Δ is odd, there can be one additional perfect matching). A graph is *Hamilton-connected* (HC) if for any distinct vertices, there is an HP from one to the other.

Finally, ILP refers to integer-linear programming: the solving of linear optimization problems with linear constraints, but with solutions restricted to integers. Many graph theory problems are easy to set up as ILP problems, using 0-1 valued variables, and there are several software packages (I use *Mathematica*) that allow solution of modest-sized ILP problems.

The triangular grid graph

For the triangular grid graph interesting parameters are α and θ. The first is known [G]: $\alpha(TG_n) = \lfloor V/3 \rfloor$, where V is $n(n+1)/2$. For θ, the question is how many triangles are needed to cover the graph; that was the subject of an investigation by Conway and Lagarias, who viewed it as a tiling problem. When $n \equiv 0, 2, 9, 11 \pmod{12}$, V is divisible by 3 and $\theta = V/3$ ([CL, p. 197, by induction]). But when $n \equiv 3, 5, 6, 8 \pmod{12}$, V is divisible by 3 but $\theta \neq V/3$. This negative result is the heart of [CL]; see also [WE]. In these cases one can easily succeed with one extra triangle by starting with one of the positive cases and covering the additional rows in a simple way. Working mod 12, one can use the 0-case to get 3, 2 to get 5 and 6, and 5 to get 8. Similar arguments from the positive cases show that when $n \equiv 1, 4, 7, 10 \pmod{12}$, θ is the minimum possible, $\theta = (V+2)/3$.

Editor note: Normally I would put some diagrams here illustrating the

proof above, but space is short and it is fairly easy following path of least resistance.

Getting $\chi = 3$ for these graphs ($n \geq 2$) is easy, as is $\chi' = 6$, by a method that works for traditional kings too: just use 6 colors, 2 in each direction alternating to avoid collision.

For the domination number γ, [D] presented the upper bound $7\binom{\lceil n/7 \rceil + 1}{2} - 1$. Computations using ILP up to $n = 31$ yielded the following exact values (the ones past 25 were obtained by Rob Pratt (SAS)). Beyond $n = 14$ (see Fig. 2) the numbers follow a succinct formula, which we state as a conjecture

1, 1, 2, 3, 3, 5, 6, 7, 9, 10, 13, 15, 17, 19, 21, 24, 27, 30, 33, 36, 40
43, 47, 51, 55, 59, 63, 68, 72, 77, 82

Conjecture. For $n \geq 14$, $\gamma(TG_n) = \left\lfloor \frac{n^2+7n-23}{14} \right\rfloor$

As noted in [D] this is related to covering the board with the 7-sized sets hexagon-plus-neighbor-hexagons. Issues arise because of the boundary, so there is unlikely to be a universal formula for $\gamma(TG_n)$.

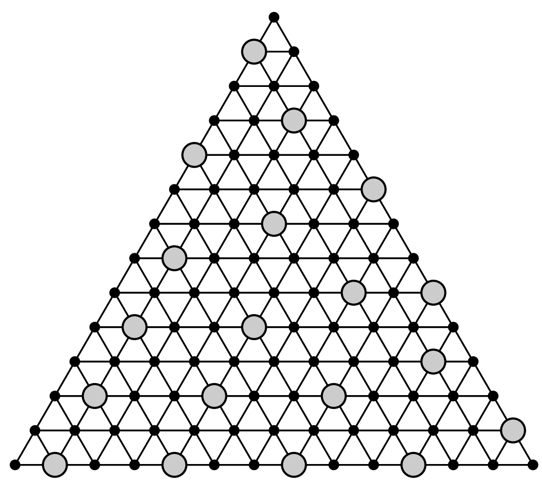

Figure 2. A minimal dominating set of size 19 for TG_{14}.

Hex rooks

The rook in hex chess moves along rows of adjacent cells in any direction. The graph (called a queen in [D]), denoted HR_n, is shown in Figure 3.

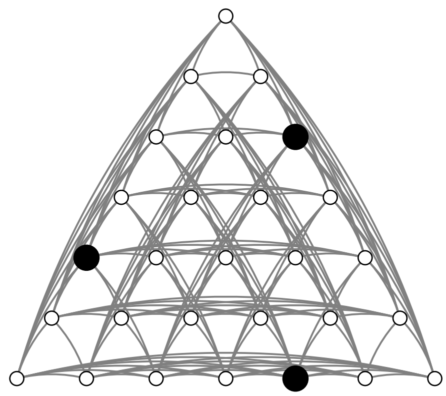

Figure 3. The hex rook graph, HR_7, shown with a dominating set of size 3.

Some parameters have been studied. The independence number is known: $\alpha(HR_n) = \lfloor (2n+1)/3 \rfloor$ (Harborth; see [KVW, prob. 131]). And the chromatic number is an amusing puzzle: $\chi(HR_n) = n$ except for $n = 2$ and 4, in which case $\chi = n + 1$ [U]. The clique covering number θ is not hard to resolve. The lower bound proof here is due to Dan Ullman.

Proposition. For $n \geq 2$, $\theta(HR_n) = n - 1$.

Proof. Use induction starting from the easy base cases of 2 and 3. Adding the $\wedge$-shape formed by two K_n's joined at the apex yields $\theta(HR_n) \leq \theta(HR_{n-2}) + 2 = (n-3) + 2 = n - 1$. For the lower bound, suppose HR_n is covered with $n - 2$ cliques. Then the cliques cover the $3n - 3$ boundary vertices, and because $3(n-2) < 3n - 3$, one of the cliques has size 4 or more. But such a clique is a straight line, and therefore it must be one of the boundary n-cliques. Removing it leads to an $n - 3$ covering of HR_{n-1}, contradicting the inductive hypothesis. □

The domination number is interesting: what happens is somewhat analogous to domination for traditional chess queens. If $\gamma(HR_n) = k$, then it is easy to see that $\gamma(HR_{n+2}) \leq k + 1$: just add in the new apex, which dominates the two new slanted borders. But every once in a while a surprise occurs. The sequence starts out 1, 1, 2, 2, 3, 3, but HR_7 is a mild surprise, as there $\gamma = 7$ (Fig. 3). Moving up from there, ILP methods confirm that the sequence extends from $n = 7$ through 19 as
3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9, but then $\gamma(HR_{20})$ turns out to be 9, as found by Rob Pratt using ILP methods (Fig. 4). This means that for $n \geq 19$, $\gamma(HR_n) \leq \lfloor (n-1)/2 \rfloor$, but there are likely more surprises.

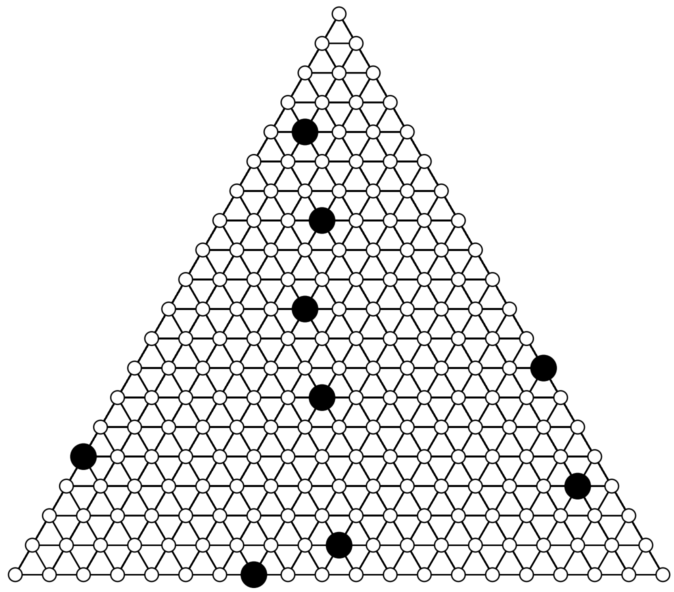

Figure 4. A size-9 dominating set for HR_{20}.

Question. Is there $n \geq 21$ for which $\gamma(\mathrm{HR}_n) < \lfloor (n-1)/2$?

Edge coloring leads to an interesting problem. The graph is regular of degree $\Delta = 2\,n - 2$. A d-regular graph with $2\,k + 1$ vertices is never class 1 (any color set has k edges at most, so d of them cover at most $d\,k$ edges, while the edge count is $d(2\,k + 1)/2 > d\,k$). Computations using an iterative coloring method based on Kempe switches show that when V is even — $n \equiv 0 \text{ or } 3 \pmod 4$ — the graph is class 1, i.e., Δ-edge colorable; I verified this up to $n = 32$. So we have the following plausible conjecture.

Conjecture. HR_n is $(2\,n - 2)$-edge colorable when $n \equiv 0 \text{ or } 3 \pmod 4$.

A proof of this will likely require a clever way of combining known edge colorings for complete graphs ($\chi'(K_n) = n - 1$ if n is even, and n if n is odd). Possibly it is worth first looking at the special case of the subgraph made from the $3\,n - 3$ boundary vertices.

More seems to be true. For $n \leq 15$, the graph has a Hamiltonian decomposition. For V even, such a decomposition gives a class-1 coloring by just splitting each Hamiltonian cycle using alternate edges. This this conjecture implies the preceding one.

Conjecture. HR_n always admits a Hamiltonian decomposition.

Even more is true in some small cases: a *perfect 1-factorization* of a regular graph with even vertex count is a class-1 edge coloring so that *any* two colors combine to form a Hamiltonian cycle. Such exist for HR_3 and HR_4

(Fig. 5), but these objects are quite difficult to study in general (their existence is an open question even for complete graphs).

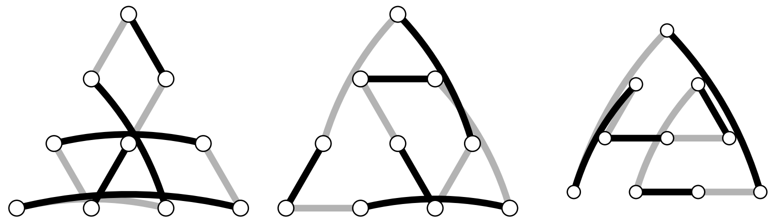

Figure 5. A perfect 1-factorization of HR_4. Any two of the six matchings shown combine to make a Hamiltonian cycle.

One can look at the fractional chromatic number (see [SU]). When $\chi(G)$ is the size of the largest clique, then $\chi_{\text{frac}}(G) = \chi(G)$. For the rook family this leaves only one case unsettled; computation shows that $\chi_{\text{frac}}(HR_4) = 13/3$.

Bishops

Let HB_n be the graph of moves of a bishop as defined in [D]: it moves along two angled rows (Fig. 1). Let $B_{m,n}$ be the graph of moves of a traditional white bishop on an $m \times n$ board with a white square at lower left. Because the latter are included in *Mathematica*'s `GraphData` database, and because there are good algorithms to determine graph isomorphism, one learns quickly, and surprisingly, that HB_n is isomorphic to $B_{n,n+1}$ (Proof omitted, but note that these two graphs have the same edge count.) So problems about *HB* graphs reduce to questions about traditional bishops.

The classic problems of α and θ for bishops have nice formulas. For domination, γ for the full (2-component) bishop graph on an $n \times n$ board is n, as proved in [C]. For our white bishop graph, this becomes $\gamma(B_{n,n}) = \lfloor n/4 \rfloor + \lceil n/4 \rceil$. The dominating set is just the column nearest the center. The problem is perhaps still open for the rectangular case.

More recently, Berghammer [B] showed that

$$\alpha(B_{m,n}) = \begin{cases} n-1 & \text{if } m = n \\ \frac{m+n}{2} - 1 & \text{if } m \neq n \text{ and both are even} \\ \left\lfloor \frac{m+n}{2} \right\rfloor & \text{if } m \neq n \text{ and one or both of } m,\ n \text{ is odd} \end{cases}$$

A typical maximum independent set is shown in Figure 6.

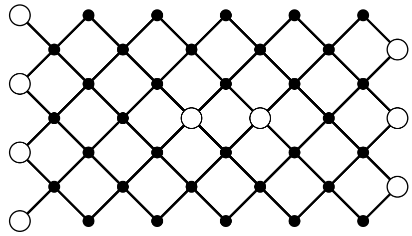

Figure 6. $B_{7,12}$, with a typical maximum independent set

Berghammer's work focuses on the full bishop graph, but yields the results shown for the white component. In [H] the problem is investigated in the context of a board based on a torus, Möbius strip, or cylinder.

Peter Saltzmann and I [S] proved that $B_{m,n}$ is always class 1 and has a Hamiltonian cycle except for $B_{1,n}$, $B_{2,n}$, and $B_{3,3}$. Computation suggested that $B_{m,n}$ is generally HC; a proof is given here. The related problem of when the traditional knight graph is Hamilton-laceable (i.e., there is a knight tour from any white square to any black square on a standard board) was recently settled, again with the path to the result based on computation: the knight graph for an $m \times n$ board is Hamilton-laceable iff $m \geq 6$, $n \geq 6$, and one of m, n is even [DW].

Theorem. The bishop graph $B_{m,n}$ (assume $m \leq n$) is Hamilton-connected iff $(m, n) \in \{\{1, 1\}, (2, 2)\}$ or $m \geq 4$ and $n \geq 5$.

Proof. The graph is disconnected when $m = 1$ and a path when $m = 2$, so is not HC in these cases except for the trivialities of $B_{1,1}$ ($\cong K_1$) and $B_{2,2}$. ($\cong K_2$). The graph $B_{3,n}$ is not HC: it is easy to see there can be no HP from the lowest left vertex in the center row to the one up and right of it (*A* and *B* in Fig. 7, right). Similarly, there is no path from *A* to *B* in $B_{4,4}$ (Fig. 7, left). So it remains to show that the rest are HC.

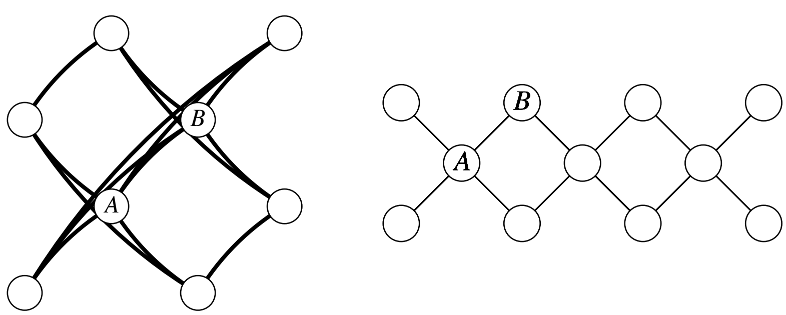

Figure 7. In both of these bishop graphs, $B_{4,4}$ and $B_{3,n}$, there is no Hamiltonian path from *A* to *B*.

Computation (see [DW] for details on how to reduce the HC problem to one of finding Hamiltonian cycles) settles it for the 17 cases $B_{4,n}$ ($5 \leq n \leq 9$) and

$B_{m,n}$ ($5 \le m \le 7$ and $m \le n \le m + 3$). We then work up inductively. Consider first the sequence $B_{4,5}$, $B_{4,10}$, $B_{4,15}$,…, as illustrated in Figure 8.

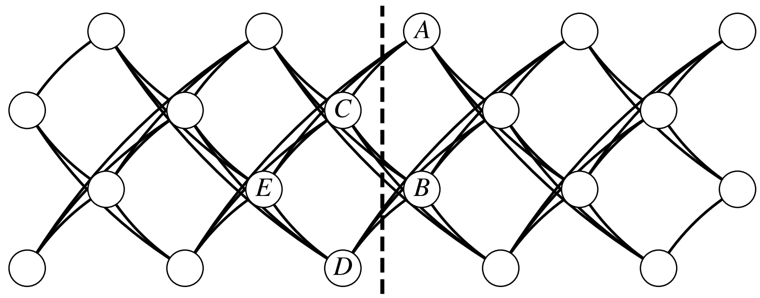

Figure 8. The graph $B_{4,10}$ breaks into two copies of $B_{4,5}$.

The two halves of the graph are HC; think of the leftmost $B_{4,5}$ as the old graph, and the right half as the new. Now consider a source-target pair (S, T).

Case 1. S is new and T is old: Take an HP in the new part from S to B, then go to C (or to D, if $C = T$) and use an HP in the old part to get to T. Proceed similarly If $S = B$: use a new HP from B to A and then go to C and on to T by an old HP, but use E instead of C if $T = C$.

Case 2. S and T are both new: By switching S and T, we may assume $S \ne B$. Find an HP in the new graph from S to T. At some point it has a segment XBY (or just XB if $T = B$). Depending on the locations of X and Y, use an HP in the old graph to create a detour of the form $XBC\cdots DY$ or $XBD\cdots CY$. If the path ends in XB, use the detour $XC\ldots DB$ or $XD\ldots XB$.

Case 3. S and T are both old. An argument symmetric to that of Case 2 works.

The preceding construction works to extend the HC property from $B_{4,n}$ to $B_{4,n+5}$. Since we can start with n being any of 5, 6, 7, 8, 9, this gets the result for $B_{4,n}$, with $n \ge 5$. The same idea works for the families $B_{m,n}$, where $5 \le m \le 7$. It is slightly simpler because in the smallest cases the new part is $B_{5,4}$ which is the same as $B_{4,5}$, and so HC. With these four infinite rows done, we can move up by the same technique. To get that $B_{m,4k+i}$ is HC, work up in steps of 4 from the case $B_{m,4+i}$; for example, the first case is $B_{8,9}$ which breaks down into $B_{4,9}$ and $B_{4,9}$, both of which are HC. □

Traditional queens

There are many well-studied problems for queens on a traditional $m \times n$ board, $Q_{m,n}$, where we assume $m \le n$. The independence number is known for $Q_{m,n}$: it is min (m, n) except for $Q_{2,2}$ and $Q_{3,3}$. There appear to be no patterns to the χ and γ sequences [O]). But χ' shows a nice pattern. W. Jarnicki, J. DeVincentis, and I proved the following.

Theorem. The graph $Q_{m,n}$ is class 1 if at least one of m and n is even, or if m and n are equal and odd. The graph is class 2 if m and n are odd and $n \geq (2m^3 - 11m + 18)/3$.

Proof. We will prove only the class 2 result here. First one needs the maximum degree and edge count of the graph. Some counting and summations yield

$\Delta(Q_{m,n})$ is $3m + n - 5$ if $m = n$ and n is even and $3m + n - 4$ otherwise;

$$E(Q_{m,n}) = \frac{1}{6} m \left(2 - 2m^2 - 12n + 9mn + 3n^2\right)$$

If m and n are odd, then ρ, the number of edges in the largest matching in $Q_{m,n}$, is at most $(mn - 1)/2$ vertices. Combining this with the formulas for Δ and E and the hypothesized inequality leads to $\Delta \cdot \rho < E$, so the edges cannot be colored in Δ colors. □

Conjecture. The queen graph $Q_{m,n}$ is class 2 iff m and n are odd and $n \geq (2m^3 - 11m + 18)/3$.

Computations show that the conjecture is true for $m = 3$ or 5. The case of 5 requires finding class 1 colorings for $Q_{5,n}$ where $n = 5, 7, \ldots, 69$. The last two required some extensions to my iterative algorithm. For $n = 67$ I had to start with a precoloring of a subgraph isomorphic to $Q_{5,65}$. And similarly, $n = 69$ succeeded only when I precolored a $Q_{5,67}$ subgraph.

Conclusion

Chess pieces of various sorts give rise to intriguing graphs and studying their properties can yield nice conjectures, and sometimes simple proofs. It is clear that having powerful algorithms ready to generate data is critical to discovering the patterns.

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