Circle-Squaring: A Mechanical View

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1. Circle-Squaring Variations

There are several variations of the classic circle-squaring problem to keep modern circle-squarers busy. Laczkovich's proof that a disk can be taken apart into finitely many disjoint pieces that can be reassembled to a square (necessarily of the same area) was a tour-de-force of modern mathematics (see [GW]). A much more elementary connection between the circle and the square is the fact, found by G. B. Robison in 1960 (see [W]), that a square can serve as a perfectly good wheel (i.e., roll smoothly with no vertical shift in the center of gravity) on a road of linked inverted catenaries. The goal of this paper is to investigate how one can turn circular motion into square motion by a purely mechanical linkage; an application is to the construction of a drill that drills *exact* square holes. We conclude with an extension of this idea to a drill that drills hexagonal holes.

A well-known construction that comes close is due to James Watts, who in 1914 had the idea of rotating a Reuleaux triangle within a square (see [G, pp. 36–37]). He started a company, Watts Brothers Tool Works in Wilmerding, Pennsylvania, that is still in operation and makes and sells square-hole drills based on this idea. A Reuleaux triangle is a shape made from arcs of circles centered at the vertices of an equilateral triangle (Fig. 1). It has constant width. When rotated inside a square, each vertex traces a curve that is almost a square. If one makes a cutting tool at each vertex (by cutting away part of the device so as to have a sharp end at each vertex) then this shape can be used to make a working drill that drills almost-square holes. For this to work from a rotating drive (such as a drill press) one must force the Reuleaux triangle to rotate inside the square, and that requires a square template to constrain the Reuleaux triangle as well as a special coupling to address the fact that the center of rotation moves. In this paper we will discuss an idea that is no more complex, but leads to a drill that produces *exact* square holes.

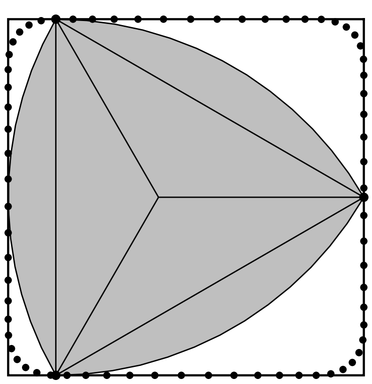

Figure 1. The locus of a vertex of the Reuleaux triangle is almost a square.

Note that the most obvious way of squaring circular motion doesn't quite work. One could imagine rolling a disk in the space between two concentric squares (Fig. 2). The locus of the center of the disk will be a near-square.

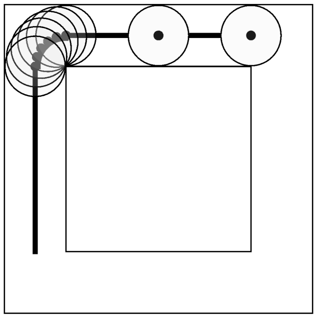

Figure 2. When a disk rolls between two squares its center does not trace out an exact square.

Another option is to slide a square in the space between two squares. The center of the sliding square would then trace out a perfect square. And the sliding could be effected by turning a crank; one can imagine a rod extending from the center of the device and containing the center of the small square in such a way that the center freely slides along the rod. Then as the rod rotated, the small square would be forced through the annulus. But there would be a problem related to the speed of the square's motion. If the rotating crank moved at constant angular speed, the small square would smash into the dead end, just before it has to turn downward. So while the geometry works, the motion would be not at all smooth. Since the idea is a device that works when the driving rotation is at constant speed, we seek a much smoother motion. Mathematically, we want is a linkage that takes standard uniform circular motion and turns it into square motion in a differentiable way.

2. The Rotor for the Square

A mechanical device that does the job is nicely presented in the book by Bryant and Sangwin [1], who resurrect an idea they found in an anonymous 1939 article in *Mechanical World*. The main point is to use a variation of the classic Reuleaux triangle. The variation is shown in Figure 3, where the starting point is the isosceles right triangle ABC with hypotenuse AB of length 2. An arc of radius $\sqrt{2}$ centered at C forms the bottom border and the circle centered at C and having radius $2 - \sqrt{2}$ defines, via its topmost 90° sector, the upper border. The right and left borders simply connect the top and bottom via arcs of radius 2 centered at A and B, respectively, as in the classic case. Call this Reuleaux variation the *rotor*.

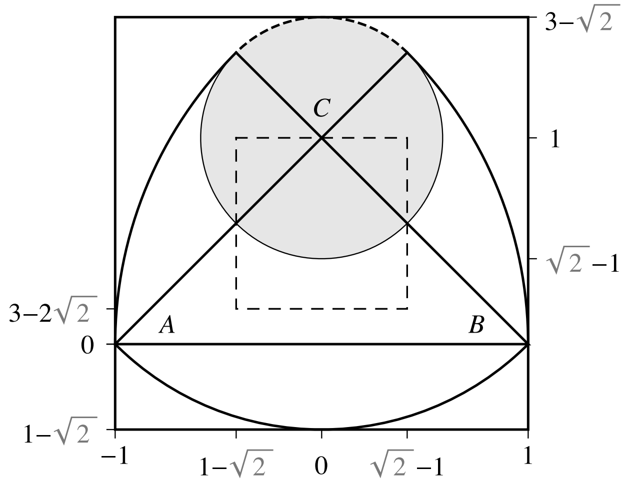

Figure 3. A variation of the classic Reuleaux triangle; the inner square is the locus of C.

The key properties of the rotor are as follows; the proofs are simple and left as an exercise.

• The rotor is a curve of constant width 2.

• If the rotor rotates so that it always lies within the square of side-length 2, then C traces out an exact square of side-length $2\sqrt{2} - 2$.

In order to generate an animation (Fig. 4 and [W1]) one must first understand the translation needed to return the rotor to the square after the rotation. The point C starts at (0, 1). Suppose the rotor has rotated so that the circle centered at C has rotated counterclockwise through θ radians. Then the complete transformation consists of this rotation around center C followed by a translation to bring the rotor back into the square. The details are straightforward and the exact translation vector, call it $\boldsymbol{\tau} = (\tau_x, \tau_y)$, is defined in eight cases, depending on θ, as follows.

$$\boldsymbol{\tau} = \begin{cases} (1 - \cos\theta - \sin\theta, 0) & \text{if } 0 \le \theta \le \pi/4 \\ (1 - \sqrt{2}, \cos\theta + \sin\theta - \sqrt{2}) & \text{if } \pi/4 \le \theta \le \pi/2 \\ (1 - \sqrt{2}, \cos\theta - \sin\theta + 2 - \sqrt{2}) & \text{if } \pi/2 \le \theta \le 3\pi/4 \\ (\cos\theta - \sin\theta + 1, 2 - 2\sqrt{2}) & \text{if } 3\pi/4 \le \theta \le \pi \\ (-\cos\theta - \sin\theta - 1, 2 - 2\sqrt{2}) & \text{if } \pi \le \theta \le 5\pi/4 \\ (\sqrt{2} - 1, \cos\theta + \sin\theta + 2 - \sqrt{2}) & \text{if } 5\pi/4 \le \theta \le 3\pi/2 \\ (\sqrt{2} - 1, \cos\theta - \sin\theta - \sqrt{2}) & \text{if } 3\pi/2 \le \theta \le 7\pi/4 \\ (\cos\theta - \sin\theta - 1, 0) & \text{if } 7\pi/4 \le \theta \le 2\pi \end{cases}$$

Each of these expressions is constant in one coordinate, which essentially proves that C traces a square.

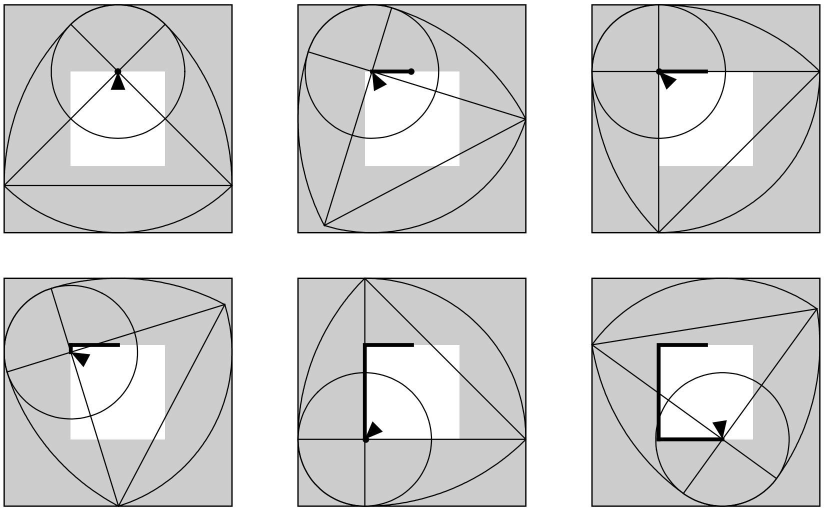

Figure 4. Images from the rotation of the rotor inside a square.

If one places a cutting tool at point *C* (indicated by the black triangle in Fig. 4) and turns the rotor so that it stays inside the large square, then *C* traces out an exact square. Thus the device can be viewed as a drill that drills an exact square hole, though we need to bring the construction into the third dimension to get a working model.

Another benefit of the formula for the transformation is that we can look at the speed of the point *C* as it traverses the square, an important practical consideration. Figure 5 shows the graph of the speed: the magnitude of the velocity vector of the point generating the square locus. The speed is 0 at the four corners of the small square, meaning that the point naturally slows to a stop at these points. This is visually evident from an animation [W1, W2] and is critical in the construction of a device that works well when the crank is turned at a uniform rate. The graph in Figure 5 is made up of straight-looking segments, but they are not exactly straight.

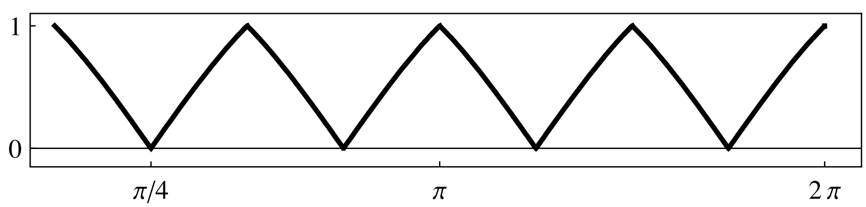

Figure 5. The speed of the cutting point as it follows the square locus.

3. Moving to Three Dimensions

Now that we have a rotation method that produces the square, we can use an Oldham coupling to describe a device that works in three dimensions from pure circular motion. The practical importance of this enhancement is that the driving end can be placed in a standard drill press; the other end, when restricted to stay inside the ambient square, will yield a perfectly square locus for point *C*, and this can be turned into a working square-hole drill. A video of such a drill in action can be seen at [BS]; enter the site and click on Applications of Non-Roundness. But the theoretical importance is just as interesting: this enhancement gives us a device that goes smoothly from circular motion to perfect square motion.

The key to making this work is the Oldham coupling, which usually refers to the use of interlocked sliders to translate rotational motion from one center of rotation to another (see [Kab]). However, it can also be used to go from a fixed rotational center to a variable one. For a physical model, one need not worry about the equations: they are solved physically as the coupling is forced by the square surrounding the rotor to work as desired. But for a computer model one must work out the equations of motion.

The coupling consists of two bars attached at their centers, so that they form a large cross; see Figure 6, which shows the rotor in its housing and a driving disk with a crank. One half of the coupling slides in the slot in the protrusion of the disk in the rotor; the other slides in a similar indentation in the driving disk. The bars of the coupling are free to slide and that is why, in the physical device, the driving mechanism can be simple rotation while the business end rotates and translates so that it stays inside the ambient square. In the right-hand image of Figure 6 the value of θ is 5.9 and the almost completed square that would be traced out by the cutting tool is shown.

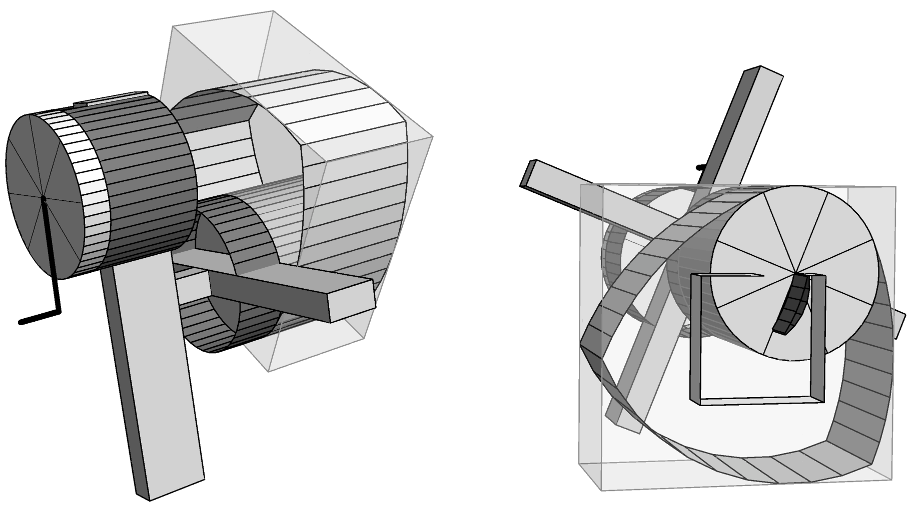

Figure 6. Two views of the entire device. The crank at left moves in standard circular motion and the cutting tool attached to the center of the circle on the other end traces out an exact square.

To make these diagrams and the corresponding computer animation one needs to understand the mathematics of the couplng in detail; that is, one must determine the locus, as the rotation proceeds, of the center of the Oldham cross so that the two rods can be located to fit properly into the rotated slots with the rotor properly positioned in its ambient square.

We can ignore the third dimension and work in a plane orthogonal to the rotating axis. Referring to Figure 7, which shows the Oldham bars as thick gray lines and the fixed driving circle as dashed, let θ be the amount the crank has rotated counterclockwise, 120° in the figure. The Oldham center has the form $(0, 1) + t(-\sin\theta, \cos\theta)$ because it lies on the bar through $(0, 1)$. But it also lies on the perpendicular bar and so, recalling the vector $\boldsymbol{\tau}$ from §2 that gives the translation of the rotor after rotation, it has the form $(0, 1) + \boldsymbol{\tau} + s\,(\cos\theta, \sin\theta)$. Setting these two forms equal and solving leads to the following expression for the Oldham center:

$$\frac{1}{2}\left(\tau_x\,(1-\cos(2\,\theta)) - \tau_y\sin(2\,\theta),\ 2 + \tau_y\,(1+\cos(2\,\theta)) - \tau_x\sin(2\,\theta)\right)$$

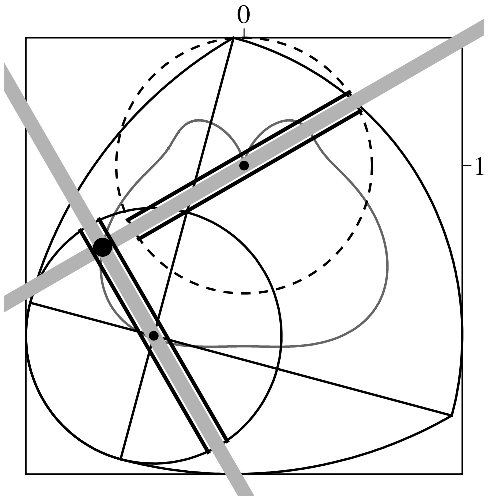

Figure 7. The motion of the Oldham coupling — the gray bars free to slide in the slots in the circles — can be understood by viewing everything in a plane.

So now we have the formulas needed to make an animation of the complete device (and generate the diagrams in Figs. 6 and 7); the animation can be viewed at [W2]. This work allows us to see the locus of the Oldham center, the intriguing heart-shaped curve shown in gray in Figure 7.

To compare this method with the classic Watts construction note that, like Watts, it has a square template to keep the rotor in the right place. And for both methods the Oldham coupling handles the variable center of rotation. So this device has the same complexity as that of Watts, but by using a four-sided rotor it produces an exact square.

4. Hexagonal Complications

A natural question is whether this construction generalizes to regular n-gons. We do not know about the case of n odd, but the square ideas do point the way when n is even, and we present here the details of the hexagonal case, which presented some unexpected difficulties.

One starts with a circle of radius r centered at $O = (0, 1)$, the point that will trace out an exact hexagon. Draw diagonals connecting O to C and D and spanning a $\pi/3$ angle (see Fig. 8) and align a unit hexagon with the circle's top. Draw the lower arc with center O and radius $\sqrt{3} - r$; it touches the bottom of the hexagon, again spanning $\pi/3$. We want this arc and the southeast side of the hexagon to meet at a point, call it A, along the diagonal from D. Solving the equations for this yields a unique value of r that works. In fact, the radius has a simple expression for any n-gon with an even number of sides:
$2\cos(\pi/n)/(2+\sec(\pi/n))$, or $3\left(\sqrt{3}-1\right)/4$ for the hexagon.

Next one draws the arc of radius $\sqrt{3}$ centered at A, starting at D, and extending to tangency at point E, and the same to get arc CB on the other side, centering at F, the reflection of A.

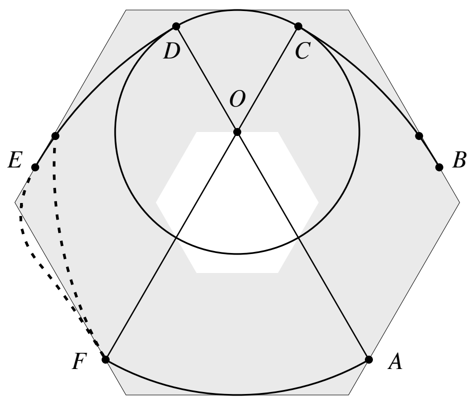

Figure 8. A failed attempt to make a hexagon driller.

But now there are two problems. It is natural to try the arc from B that passes through F (dotted circular arc in Fig. 8). This comes nowhere near point E. But there is a bigger problem. First, we need to define the main transformation T_θ, which consists of rotating the rotor counterclockwise around O through θ and then translating so that A, the device's anchor, is returned to the hexagon. This definition applies when $0 \leq \theta < \pi/6$, but it extends by rotational symmetry to the full range of θ. For this initial part of the θ-domain $T_\theta(P)$ is

$$R_\theta^O(P) + \left(\tfrac{1}{4}\left(3+\sqrt{3} - \left(3+\sqrt{3}\right)\cos\theta - \left(1+\sqrt{3}\right)\sin\theta\right]\right),$$

where R_θ^O denotes rotation around O. The problem that surfaces is that when $0.24 < \theta < \pi/3$, the point $T_\theta(E)$ is outside the hexagon (see the dashed locus of E in Fig. 8).

The diagram suggests the solution of just terminating the arc rising from F at the point, call it E_1, where it strikes arc DE, and then truncating DE below E_1. This rotor appears to work, but does not. When $0 < \theta < 0.049$ there is not enough contact with the hexagon to guarantee rigidity of the rotor within the hexagon. This is because E_1's position will be inside the hexagon and the transformed arc near F does not touch the southwest side.

A correct construction is found by fixing the rotor and letting the outer hexagon rotate around it. The shape of the curve joining point F to arc DE is then determined from the envelope formed by the southwest side of the hexagon as the hexagon rotates clockwise through an angle of $\pi/6$. To simplify things we take the hexagon as depicted in Figure 9, ignoring three of its sides. We then extend the remaining sides to form an equilateral triangle and rotate that triangle clockwise so that the southeast side passes through A and the north side remains tangent to arc CD.

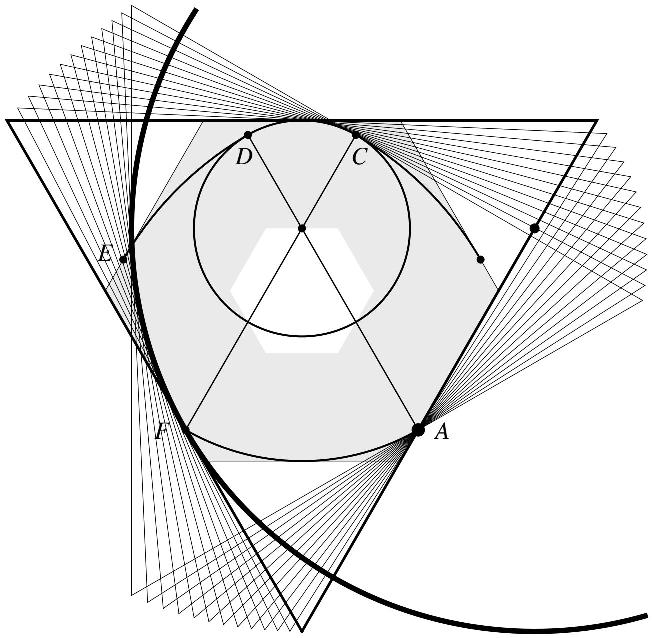

Figure 9. The construction used to fill in the missing link on the southwest side of the rotor: it is an envelope of a rotated triangle, whose initial position is thickened. The envelope turns out to be a circular arc.

The method of finding the envelope is to determine a formula for the southwest side of the triangle in terms of the the rotating angle θ. That is, we consider the family of lines as $(X(\theta, t), Y(\theta, t))$, where t is the "running parameter" on each line. Now the envelope we seek defines a relationship $t = f(\theta)$ and so it may be viewed as a parametric curve in θ via $(X(\theta, f(\theta)), Y(\theta, f(\theta)))$. We can determine f by computing slopes in two ways.

First, the slope on the envelope at some fixed θ is $Y_t(\theta, f(\theta)) / X_t(\theta, f(\theta))$, the slope of the line that is tangent to the envelope at the given point. But the slope is also $\frac{d}{d\theta} Y(\theta, f(\theta)) \Big/ \frac{d}{d\theta} X(\theta, f(\theta))$ and the chain rule applies yielding the relationship

$$\frac{Y_\theta + Y_t f'(\theta)}{Y_\theta + Y_t f'(\theta)} = \frac{Y_t}{X_t}$$

Cross-multiplying gives the nicely concise envelope equation $Y_\theta X_t = Y_t X_\theta$. Since we know X and Y explicitly — they are obtained by inverting T_θ on the parametrized enveloping line, specifically,

$$(X, Y) = R^O_{-\theta}\left(-\frac{1}{2}, \frac{1}{4}\left(1 - 3\sqrt{3}\right) + \frac{\sqrt{3}}{2} t\right) - \left(\frac{1}{4}\left(3 + \sqrt{3} - \left(3 + \sqrt{3}\right)\cos\theta - \left(1 + \sqrt{3}\right)\sin\theta\right], 0\right)$$

— this relationship becomes one that can be solved for t in terms of θ. The preceding formula expands to

$$\frac{1}{4}\left(\left(\sqrt{3} + 3\right)\cos^2\theta + \left(-2t - \sqrt{3} + \sin\theta\left(\sqrt{3} + 1\right) - 3\right)\cos\theta + \left(2\sqrt{3}\,t - 3\left(\sqrt{3} + 1\right)\right)\sin\theta,\right.$$
$$\left.\left(4 + \left(2t + \sqrt{3} - \left(\sqrt{3} + 1\right)\sin\theta + 3\right)\sin\theta + \left(2t\sqrt{3} - 3\left(\sqrt{3} + 1\right) - \left(\sqrt{3} + 3\right)\sin\theta\right)\cos\theta\right)\right)$$

We can now differentiate and solve the envelope equation algebraically to determine that $t = f(\theta) = \frac{1}{4}\left(3 + \sqrt{3} + 2\left(1 + \sqrt{3}\right)\sin\theta\right)$ and that in turn gives the envelope as a function of θ. To our pleasant surprise the envelope is the arc of a circle of radius $\frac{3}{4}\left(1+\sqrt{3}\right)$ centered at $X=\left(\frac{1}{4}\left(3+\sqrt{3}\right),\,1\right)$.

Knowing that the curve we seek is a circle leads to another, more direct approach. Consider the line through F perpendicular to S, the southwest side of the hexagon, and search for a point X on this line so that $T_\theta(X)$ has a constant distance from S (i.e., its motion is parallel to S). If such an X exists, then the circle centered at X and passing through F will stay tangent to S through the relevant part of the transformation. Setting up the equations is simple using T_θ and a parametric representation of the perpendicular line; the result is that the point $X=\left(\frac{1}{4}\left(3+\sqrt{3}\right),\,1\right)$ does the job (Fig. 10). Its distance from S — the radius of the arc — is $\frac{3}{4}\left(1+\sqrt{3}\right)$. We run this arc to where it strikes arc DE and, using symmetry to flip X to Y and connect A to B_2, this completes the construction.

The complete rotor made from six circular arcs is shown in Figure 10. The point E_2 is not on the hexagon but F is and that forces rigidity, since there can be no vertical motion and points A and F prevent horizontal motion. For the next little bit of rotation the arc above F combines with A to prevent left-right motion, and then E_2 reaches the hexagon at which point it combines with A to stop left-right motion. An animation that shows the set of points (either four or five) on the rotor that touch the ambient hexagon is available at [W3]. The locus of E_2 is interesting: in one full rotation it strikes the outer hexagon 12 times (Fig. 10).

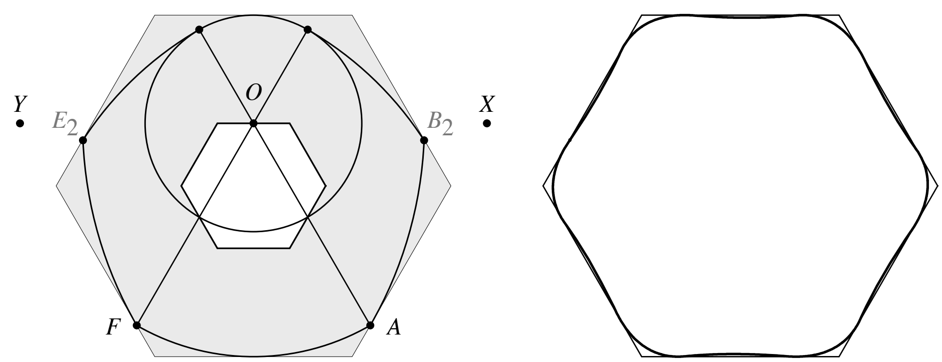

Figure 10. A correct roller for getting a perfect hexagonal locus. It is made up of six circular arcs, centered at O, A, X, O, Y, F. The image at right shows the locus of E_2 as θ runs from 0 to 2π.

Similar ideas work for the octagon, and it seems likely that they will extend to regular n-gons when n is even. So the main unresolved problem is whether one can construct a device along these lines that will make a 3- or 5-sided hole.

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References

[BS] John Bryant and Chris Sangwin, *How Round Is Your Circle?,* Princeton University Press, 2008; associated website: http://HowRound.com

[G] Martin Gardner, *The Colossal Book of Mathematics*, W. W. Norton, New York, 2001.

[GW] Richard Gardner and Stan Wagon, At long last, the circle has been squared, *Notices of the American Mathematical Society,* **36** (1989) 1338-1343.

[Kab] Sandor Kabai, Oldham coupling, The Wolfram Demonstrations Project, http://demonstrations.wolfram.com/OldhamCoupling.

[W] Stan Wagon, The ultimate flat tire, *Math Horizons* (Feb. 1999) 14-17.

[W1] Stan Wagon, Drilling a square hole, The Wolfram Demonstrations Project, http://demonstrations.wolfram.com/DrillingASquareHole.

[W2] Stan Wagon, Square hole drill in three dimensions, The Wolfram Demonstrations Project, http://demonstrations.wolfram.com/SquareHoleDrillInThreeDimensions.

[W3] Stan Wagon, Drilling a hexagonal hole, The Wolfram Demonstrations Project, http://demonstrations.wolfram.com/DrillingAHexagonalHole.