

How to lose as little as possible

Vittorio Addona, Stan Wagon, and Herb Wilf

Abstract

Suppose Alice has a coin with heads probability q and Bob has one with heads probability $p > q$. Now each of them will toss their coin n times, and Alice will win iff she gets more heads than Bob does. Evidently the game favors Bob, but for the given p, q , what is the choice of n that maximizes Alice's chances of winning?¹ We show that there is an essentially unique value $N(q, p)$ of n that maximizes the probability $f(n)$ that the weak coin will win, and it satisfies $\left\lfloor \frac{1}{2(p-q)} - \frac{1}{2} \right\rfloor \leq N(q, p) \leq \left\lceil \frac{\max(1-p, q)}{p-q} \right\rceil$. The analysis uses the multivariate form of Zeilberger's algorithm to find an indicator function $J_n(q, p)$ such that $J > 0$ iff $n < N(q, p)$ followed by a close study of this function, which is a linear combination of two Legendre polynomials. An integration-based algorithm is given for computing $N(q, p)$.

¹The problem of determining the optimal N first appeared in [4].

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1 The problem

Suppose Alice has a coin with heads probability q and Bob has one with heads probability p . Suppose $q < p$. Now each of them will toss their coin n times, and Alice wins iff she gets more heads than Bob does (n.b.: in case of a tie, Bob wins). Evidently the game favors Bob, but for the given p, q , what is the choice of n that maximizes Alice's chances of winning?

Interestingly, there is a nontrivial (i.e., in general > 1) unique value of n that maximizes her probability of winning. For example, in the case $p = 0.2, q = 0.18$, Figure 1 is a plot of Alice's win probability as a function of n .

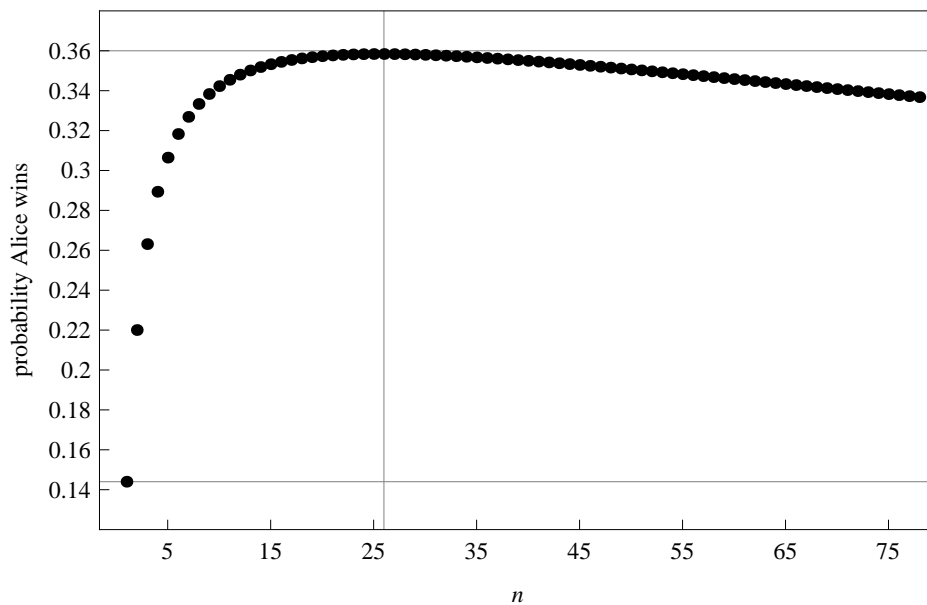


Figure 1: Probability that Alice wins vs. n .

In this example, if each player flips their coin 26 times, which is the best choice for her, Alice's chance of winning will be about 0.36, compared to a chance of 0.14 if each coin is tossed only once.

In general, her chances of winning are

$$f(n) = f(n, p, q) \stackrel{\text{def}}{=} \sum_{r \geq 0} \binom{n}{r} p^r (1-p)^{n-r} \sum_{s > r} \binom{n}{s} q^s (1-q)^{n-s}. \quad (1)$$

This problem arose from a consideration of real-world events in the National Football League, where teams play a season of 16 games and do not play all other teams. If teams A and B have probabilities q and $p > q$, respectively, of winning any game and never play each other, one can wonder about the chance that A's season record will be strictly better than that of B. That is easy to answer, but then one is led to the question of whether the season length, 16, is favorable or not to such an outcome and what the optimal choice would be.

We will also give, in section 10 an algorithm that uses repeated numerical integration to compute the optimum value $N(q, p)$. Mathematica code for various computations, graphics, and algorithms (e.g., the generation of graphs of p_n or computation of $N(q, p)$) is available in the electronic supplement at [5].

Much of the work here has relied on computing power, both for numerical experiments and for proofs using symbolic computation. Some sophisticated algorithms in Maple and Mathematica (Zeilberger's `MultiZeil`, and cylindrical algebra reduction of polynomial systems) played crucial roles; without them the discoveries and proofs would have been difficult, if not impossible, to find.

1.1 Acknowledgments

Professor Bruno Salvy, of INRIA, France, has kindly supplied to us some highly refined asymptotic results, which were of great assistance in this work. We thank Rob Knapp for some helpful discussion on the differential equation aspect of the problem.

2 Overview of methods and results

It develops that there is, in this problem, a nice *indicator function* $J_n(q, p)$, which is simply a linear combination of two consecutive Legendre polynomials, with the property that the sign of $f(n + 1) - f(n)$ is the same as the sign of $J_n(q, p)$. We will find this indicator by using the multivariate form of Zeilberger's algorithm [1]. We will then show that for small n , J is positive and for large enough n , J is negative, and that there is only a single integer value of n , or a consecutive pair $(n, n + 1)$, at which the sign of J changes. Thus f has a unique maximum, at $n = N(q, p)$, say. Here is the precise result.

Theorem 1. *With $f(n)$ defined by (1) we have*

$$\frac{f(n + 1) - f(n)}{((1 - p)(1 - q))^{n+1}} = \left(y + \frac{1}{2}(1 + xy) \right) \phi_n(xy) - \frac{1}{2}\phi_{n+1}(xy), \quad (2)$$

where $x = p/(1 - p)$, $y = q/(1 - q)$, and

$$\phi_n(z) = \sum_{r=0}^n \binom{n}{r}^2 z^r = (1 - z)^n P_n \left(\frac{1 + z}{1 - z} \right), \quad (3)$$

and $P_n(t)$ is the classical Legendre polynomial. Therefore the indicator function

$$J_n(q, p) = \left(y + \frac{1}{2}(1 + xy) \right) \phi_n(xy) - \frac{1}{2} \phi_{n+1}(xy)$$

has the desired properties.

Next, in section 5 we will prove uniqueness of and find upper and lower bounds for $N(q, p)$ by using various properties of the Legendre polynomials and by a close study of a function $p_n(q)$ which for each $q \in (0, n/(2n + 1))$, is the unique value of p for which $J_n(q, p) = 0$. The properties of the curves $p = p_n(q)$ in the (p, q) plane play crucial roles here. First, concerning uniqueness, we have

Theorem 2. (*Unimodality*) *Given probabilities $p > q$ with $p + q \neq 1$, there are either one, or two consecutive, values of n such that*

1. $f(n) \geq f(n - 1)$, and
2. $f(n + 1) \leq f(n)$, and
3. *at least one of the above two inequalities is strict.*

Definition. Given $q < p$, let $N(q, p)$ be the value of n that maximizes $f(n, p, q)$. When the value is not unique, define N to be the smaller of the two possible values that yield the maximum.

It follows from Theorem 2 and the definition of N that $N(q, p)$ is the smallest integer n such that $J_n(q, p) \leq 0$.

The resulting upper and lower bounds for $N(q, p)$ are given by

Theorem 3. *If $N(q, p)$ is the choice of n that maximizes the probability that the player with the weaker coin will win (and with ties going to the lower value) we have:*

1. $\lfloor \frac{1}{2(p-q)} - \frac{1}{2} \rfloor \leq N(q, p)$, but if $p + q \neq 1$, then $\lfloor \frac{1}{2(p-q)} + \frac{1}{2} \rfloor \leq N(q, p)$.
2. $N(q, p) \leq \left\lceil \frac{\max(1-p, q)}{p-q} \right\rceil$.

Section 9 contains proofs of various properties of the graphs of p_n , and they are used to obtain improvements to the upper and lower bounds on N .

2.1 Definitions and notation

The heads probabilities of the two coins are p and q , with $p > q$. We write $x = p/(1-p)$, $y = q/(1-q)$, $z = xy$, $u = (1+z)/(1-z) = 1 + 2pq/(1-p-q)$, $\rho = (1-p+q)/(1-p-q)$. Further, P_n is the n th Legendre polynomial and $r_n = r_n(u) = P_n(u)/P_{n-1}(u)$. The P_n 's satisfy the well known recurrence

$$P_{n+1}(u) = \frac{2n+1}{n+1}uP_n(u) - \frac{n}{n+1}P_{n-1}(u). \quad (4)$$

If we divide through by $P_n(u)$, we obtain the *ratio recurrence*

$$r_{n+1}(u) = \frac{2n+1}{n+1}u - \frac{n}{(n+1)r_n(u)}. \quad (5)$$

which will be of use in the sequel.

The *indicator function* $J_n(q, p)$, which has the sign of $f(n+1) - f(n)$, is

$$J_n(q, p) = y\phi_n(z) - \psi_n(z) = \frac{1}{2} \frac{(1-p-q)^n}{((1-p)(1-q))^{n+1}} ((1-p+q)P_n(u) - (1-p-q)P_{n+1}(u)),$$

where ϕ_n is given by (3) and²

$$\psi_n(z) = \sum_{r=0}^n \binom{n}{r+1} \binom{n}{r} z^{r+1} = \frac{1}{2}(\phi_{n+1}(z) - (1+z)\phi_n(z)) \quad (6)$$

T will denote the *interior* of the triangle in the (p, q) plane whose vertices are $(0, 0)$, $(1, 0)$, $(\frac{1}{2}, \frac{1}{2})$. T_n will be the *open* interval $(0, n/(2n+1))$. The line $L_k(q)$ in the (p, q) plane is the line $p = \frac{1}{k} + q$, and the line $M_n(q)$ is

$$M_n(q) : p = \frac{1}{n+1} + \frac{n}{n+1}q. \quad (7)$$

3 Finding the indicator function

Our first task will be to find a recurrence for $f(n)$. To do this we will use the multivariate form of Zeilberger's algorithm, `MulZeil` [1]. As usual the results that are returned by the algorithm can be easily verified by substitution.

²For a quick proof of (6), square both sides of the Pascal triangle recurrence.

Remarkably, this recurrence will show that $f(n+1) - f(n)$ is simply expressible in terms of Legendre polynomials; this will enable us to identify the values of n for which $f(n+1) \geq f(n)$ and those for which $f(n+1) \leq f(n)$.

In view of eq. (2), Alice's probability of winning increases with n as long as

$$y\phi_n(xy) - \psi_n(xy) = \left(y + \frac{1}{2}(1 + xy)\right) \phi_n(xy) - \frac{1}{2}\phi_{n+1}(xy) \quad (8)$$

is positive, and decreases otherwise. We will show that for fixed x, y there is a unique value of n at which this function changes its sign.

4 Finding the recurrence for $f(n)$

In this section we will find the recurrence that is satisfied by $f(n)$, the sum in eq. (1), using the multidimensional version of Zeilberger's algorithm. This will prove Theorem 1.

4.1 Finding the recurrence for the summand

With

$$x = p/(1 - p), \quad y = q/(1 - q), \quad g(n) = f(n)/((1 - p)^n(1 - q)^n), \quad (9)$$

the definition (1) of $f(n)$ becomes

$$g(n) = \sum_{r \geq 0} \sum_{s > r} \binom{n}{r} \binom{n}{s} x^r y^s.$$

Let $G(n, r, s) = \binom{n}{r} \binom{n}{s} x^r y^s$, be the summand. We use Zeilberger's algorithm, and his program `MulZeil` returns a recurrence

$$G(n + 1, r, s) - (x + 1)(y + 1)G(n, r, s) = (K_r - 1)(c_1(n, r, s)G(n, r, s)) + (K_s - 1)(c_2(n, r, s)G(n, r, s)), \quad (10)$$

where K_r, K_s are forward shift operators in their subscripts, and the c_i are given by

$$c_1 = c_1(n, r, s) = \frac{r(1 + y)}{r - n - 1}; \quad c_2 = c_2(n, r, s) = \frac{s(n + 1)}{(s - n - 1)(n - r + 1)}. \quad (11)$$

This is the recurrence for the summand, and it can be quickly verified by dividing through by $G(n, r, s)$, canceling all of the factorials, and noting that the resulting polynomial identity states that $0 = 0$.

4.2 Finding the recurrence for the sum

To find the recurrence for the sum, we sum the recurrence (10) over $s > r$, and then sum the result over $r \geq 0$. To do this we have first, for every function ϕ of compact support,

$$\sum_{r \geq 0} \sum_{s > r} (K_r - 1) \phi(r, s) = - \sum_{s \geq 1} \phi(0, s) + \sum_{r \geq 1} \phi(r, r), \quad (12)$$

and

$$\sum_{r \geq 0} \sum_{s > r} (K_s - 1) \phi(r, s) = - \sum_{r \geq 0} \phi(r, r + 1). \quad (13)$$

Consequently, if we sum the recurrence (10) there results

$$\begin{aligned} g(n+1) - (x+1)(y+1)g(n) &= - \sum_{s \geq 1} c_1(n, 0, s) G(n, 0, s) + \sum_{r \geq 1} c_1(n, r, r) G(n, r, r) \\ &\quad - \sum_{r \geq 0} c_2(n, r, r+1) G(n, r, r+1). \end{aligned}$$

Next we insert the values, from (11),

$$c_1(n, 0, s) = 0; \quad c_1(n, r, r) = \frac{r(1+y)}{r-n-1}; \quad c_2(n, r, r+1) = \frac{(r+1)(n+1)}{(r-n)(n-r+1)},$$

which gives

$$g(n+1) - (x+1)(y+1)g(n) = \sum_{r \geq 1} \frac{r(1+y)}{r-n-1} G(n, r, r) - \sum_{r \geq 0} \frac{(r+1)(n+1)}{(r-n)(n-r+1)} G(n, r, r+1).$$

Now substitute the values $G(n, r, r) = \binom{n}{r}^2 (u)^r$, and $G(n, r, r+1) = \binom{n}{r} \binom{n}{r+1} x^r y^{r+1}$, and simplify the result, to obtain

$$\begin{aligned} g(n+1) - (x+1)(y+1)g(n) &= \sum_{r \geq 1} \frac{r(1+y)}{r-n-1} \binom{n}{r}^2 (xy)^r \\ &\quad - \sum_{r \geq 0} \frac{(r+1)(n+1)}{(r-n)(n-r+1)} \binom{n}{r} \binom{n}{r+1} x^r y^{r+1} \\ &= -(y+1) \sum_{r \geq 0} \binom{n}{r+1} \binom{n}{r} x^{r+1} y^{r+1} + \sum_{r \geq 0} \binom{n+1}{r} \binom{n}{r} x^r y^{r+1} \\ &= y\phi_n(xy) - \psi_n(xy), \end{aligned}$$

Next, replace $g(n)$ by $f(n)/((1-p)^n(1-q)^n)$, noting that $(x+1)(y+1) = 1/((1-p)(1-q))$, to get the final result, namely that the recurrence for $f(n)$ is

$$\frac{f(n+1) - f(n)}{((1-p)(1-q))^{n+1}} = y\phi_n(xy) - \psi_n(xy) = \frac{q}{1-q}\phi_n\left(\frac{pq}{(1-p)(1-q)}\right) - \psi_n\left(\frac{pq}{(1-p)(1-q)}\right). \quad (14)$$

It is easy to find the generating function of the sequence $\{f(n)\}_{n=0}^{\infty}$ from the recurrence (14). It is

$$\sum_{n \geq 0} f(n)t^n = \frac{1}{2(1-t)} \left(1 - \frac{1 - (1-p+q)t}{\sqrt{(1 - (1-p-q)t)^2 - 4pqt}} \right). \quad (15)$$

Corollary 1. (*Symmetry*) For all p, q we have $f(n, p, q) = f(n, 1-q, 1-p)$, and consequently for all $0 < q < p < 1$ we have

$$N(q, p) = N(1-p, 1-q). \quad (16)$$

Proof 1. For a first proof, replace p by $1-q$ and q by $1-p$ in the generating function (15), and check that it remains unchanged. \square

Proof 2. For a more earthy proof, Alice's winning of the (q, p) game means she had more heads. This is identical to Bob's having more tails. That occurs when Bob wins the tails game where he has a coin that comes up tails with probability $1-p$ and Alice has a coin that comes up tails with probability $1-q$. The probability of the latter is $f(n, 1-p, 1-q)$ while the former happens with probability $f(n, q, p)$. \square

5 Proof of the unimodality theorem

In this section we will prove Theorem 2, the unimodality theorem for the optimum value of n .

According to (8), we have $f(n+1) > f(n)$ precisely for those n such that $y\phi_n(xy) - \psi_n(xy) > 0$, i.e., as long as

$$\left(y + \frac{1}{2}(1+xy)\right)\phi_n(xy) - \frac{1}{2}\phi_{n+1}(xy) > 0,$$

or equivalently, as long as

$$\frac{\phi_{n+1}(xy)}{\phi_n(xy)} < 1 + (x+2)y,$$

or

$$(1 - xy) \frac{P_{n+1} \left(\frac{1+xy}{1-xy} \right)}{P_n \left(\frac{1+xy}{1-xy} \right)} < 1 + (x + 2)y. \quad (17)$$

First suppose that $xy < 1$, i.e., that $p + q < 1$. We claim

Theorem 4. *Fix a number $x > 1$. Then the ratios*

$$\frac{P_{n+1}(x)}{P_n(x)} \quad (n = 0, 1, 2, \dots)$$

strictly increase with n .

To prove this we need the following definition and lemma.

Definition. A function $g(t)$, defined on an interval $a \leq t \leq b$, is admissible for that interval if $g(t) \geq 0$ for all $t \in (a, b)$, and for every finite sequence $\{x_i\}$ of real numbers, not all 0, it is true that $\sum_i x_i g(t)^i$ does not vanish identically on (a, b) .

Lemma 1. *Suppose $g(t)$ is admissible for (a, b) , and $h(t) \geq 0$ for all $t \in (a, b)$. Let $\mu_n = \int_a^b g(t)^n h(t) dt$, and suppose that all $\mu_i > 0$. Then $\frac{\mu_{i+1}}{\mu_i}$ is a strictly increasing function of $i = 0, 1, 2, \dots$*

Proof. Let H be the infinite Hankel matrix $\{\mu_{i+j}\}_{i,j \geq 0}$. Consider the principal submatrix formed by the first n rows and columns of H . If x_0, x_1, \dots, x_{n-1} are arbitrary real numbers, not all zero, then the quadratic form

$$Q_n = \sum_{i,j=0}^{n-1} x_i H_{i,j} x_j = \sum_{i,j=0}^{n-1} x_i x_j \int_a^b g(t)^{i+j} h(t) dt = \int_a^b \left(\sum_{i=0}^{n-1} x_i g(t)^i \right)^2 h(t) dt,$$

is clearly positive. Hence H is a positive definite matrix, whence its 2×2 principal minors $\mu_{2i} \mu_{2i+2} - \mu_{2i+1}^2$ are all positive, i.e.,

$$\frac{\mu_1}{\mu_0} < \frac{\mu_2}{\mu_1}; \quad \frac{\mu_3}{\mu_2} < \frac{\mu_4}{\mu_3}; \quad \dots \quad (18)$$

Next replace $h(t)$ by $g(t)h(t)$. Then the sequence $\{\mu_i\}_{i \geq 0}$ is replaced by μ_{i+1} ($i \geq 0$) and the Hankel matrix H is replaced by one whose (i, j) entry is μ_{i+j+1} . We apply the conclusion (18) to this new situation and we discover that

$$\frac{\mu_2}{\mu_1} < \frac{\mu_3}{\mu_2}; \quad \frac{\mu_4}{\mu_3} < \frac{\mu_5}{\mu_4}; \quad \dots \quad (19)$$

If we combine (18) and (19) we obtain the result stated in Lemma 1. \square

To prove Theorem 4 we have the integral representation

$$P_n(x) = \frac{1}{\pi} \int_0^\pi (x + \sqrt{x^2 - 1} \cos t)^n dt \quad (20)$$

for the Legendre polynomials. We can take $g(t) = x + \sqrt{x^2 - 1} \cos t$ and $h(t) = 1$ in Lemma 1 and the conclusion of Theorem 4 follows. \square

We remark that this is the reversal of a celebrated inequality of Turán which holds inside the interval of orthogonality. Now that the ratio of the Legendre polynomials on the left side of (17) is known to be a strictly increasing function of n , we observe that when $n = 0$ the left side has the value $1 + xy$, and when $n \rightarrow \infty$, the left side approaches $(1 + \sqrt{xy})^2$, which is larger than $1 + (x + 2)y$. Hence there is a unique n for which the left side of (17) is \leq the right side, but at $n + 1$ the inequality is reversed.

The case where $xy > 1$ is reduced to the case $xy < 1$, which we have just handled, by equation (16). If $xy = 1$, i.e., if $p + q = 1$, we discuss the situation in the next section. The proof of Theorem 2 is now complete. \square

6 The interesting special case $p + q = 1$

Consider the special case in which $p + q = 1$. Then $xy = p(1 - p)/((1 - p)p) = 1$, and we can carry out the calculations analytically in full. Indeed, we now have $\phi_n(1) = \binom{2n}{n}$ and $\psi_n(1) = n \binom{2n}{n} / (n + 1)$, from which we get

$$\begin{aligned} J_n(q, 1 - q) &= \frac{q}{1 - q} \phi_n(1) - \psi_n(1) = \frac{q}{1 - q} \binom{2n}{n} - \frac{n}{n + 1} \binom{2n}{n} \\ &= \left(\frac{q}{1 - q} - \frac{n}{n + 1} \right) \binom{2n}{n}. \end{aligned} \quad (21)$$

This last vanishes iff $q/(1 - q) = n/(n + 1)$, which is $q = n/(2n + 1)$. The sign of J_n then equals that of $q - n/(2n + 1)$. This proves the following.

Lemma 2.

$$J_n \left(\frac{n}{2n + 1}, 1 - \frac{n}{2n + 1} \right) = 0.$$

Now the unimodality theorem, Theorem 2, gives an explicit formula for N on this line.

Theorem 5. (*The Diagonal Formula*) For $0 < q < 1/2$, we have

$$N(q, 1 - q) = \left\lceil \frac{q}{1 - 2q} \right\rceil.$$

Proof. By the uniqueness theorem, and the tie-breaking aspect of the definition of $N(q, p)$, $N(q, p)$ is always the least integer n such that $J_n \leq 0$. Because of the agreement of the sign of J_n and that of $q - n/(2n + 1)$, the result follows. Note that when $q = n/(2n + 1)$, the $J = 0$ condition means that there is a tie between the two values $\lceil q/(1 - 2q) \rceil$ and $\lceil q/(1 - 2q) \rceil + 1$ for the optimal choice. \square

7 A general lower bound

Theorem 6. Let $N(q, p)$ denote the optimum choice of n , i.e., the one that maximizes Alice's chance of winning. Then we have

$$N(q, p) \geq \left\lfloor \frac{1}{2(p - q)} - \frac{1}{2} \right\rfloor. \quad (22)$$

First we need the following

Lemma 3. In the trapezoid τ_n , defined by the lines $q = 0$, $p = q$, $p + q = 1$, and the inequality $p \leq 1/(2n + 1) + q$, the indicator function J_n is positive. That is, if $p \leq L_{2n+1}(q)$ and $(q, p) \in T$, then $J_n(q, p) > 0$.

Proof of the Lemma. We use induction on n . Figure 2 shows the trapezoid. Since $\tau_n \subseteq \tau_{n-1}$, the induction is valid. The case $n = 1$ follows easily from $J_1(q, p)(1 - p - q)/(2q(1 - p)) = 1 - q + p(-2 + 3q)$, $L_1 = 1/3 + q$, and $q < 1/2$. Suppose that $J_n \leq 0$. Because $p < L_{2n-1}(q)$, the induction hypothesis applies, giving $J_{n-1} > 0$. Therefore $r_{n+1} \geq \rho > r_n$. The ratio recurrence tells us that $(n + 1)r_{n+1} - (2n + 1)u + n/r_n = 0$. Therefore $(n + 1)\rho - (2n + 1)u + n/\rho < 0$, or $1 - (2n + 1)p + 2nq + q < 0$. Thus $1/(2n + 1) + q < p$, contradicting $p \leq L_{2n+1}(q)$. \square

Proof of Theorem 6. Again, suppose first that $xy < 1$, i.e., that $p + q < 1$. The theorem says that if $p \leq L_{2n+1}(q)$, then n cannot be $N(q, p)$. Therefore for $n \leq 1/(2(p - q)) - 1/2$, n is not $N(q, p)$ and $N(q, p) > 1/(2(p - q)) - 1/2$. More precisely, $N \geq \lceil 1/(2(p - q)) - 1/2 \rceil = \lfloor 1/(2(p - q)) + 1/2 \rfloor$. But on the $p + q = 1$ line, $N(q, p) = \lceil q/(1 - 2q) \rceil$, which is $\lfloor 1/(2(p - q)) - 1/2 \rfloor$. So the latter works as a lower bound in both cases. Finally, if $p + q > 1$, then the symmetry formula of Corollary 1 yields $N(q, p) = N(1 - p, 1 - q)$, a transformation that leaves the bound invariant. \square

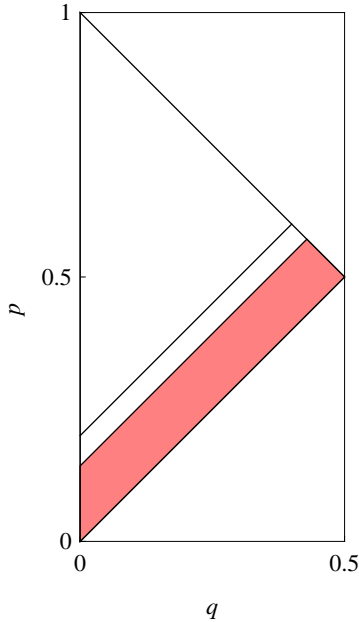


Figure 2: The trapezoid τ_n

8 The upper bound

In this section we study in detail the curves $J = 0$ and use the results to obtain a simple upper bound on $N(q, p)$ which is roughly twice the lower bound of Theorem 6.

Theorem 7. (*Upper bound on N*)

$$N(q, p) \leq \frac{\max(1 - p, q)}{p - q}.$$

8.1 Curves on which J_n vanishes

The key idea underlying our analysis of $N(q, p)$ is an understanding of the vanishing sets of J_n . Figure 3 shows these curves, together with some of the lines M_n and L_{2n+1} .

The upper black curve is $J_1 = 0$, and so on down. The blue lines are M_n and the red ones, L_{2n+1} . So we know that below the uppermost black curve $J_1 > 0$ and so $N \geq 2$. In fact, the $N = 1$ region is just the region above the first black curve. But now we need to prove various properties evident from the diagram.

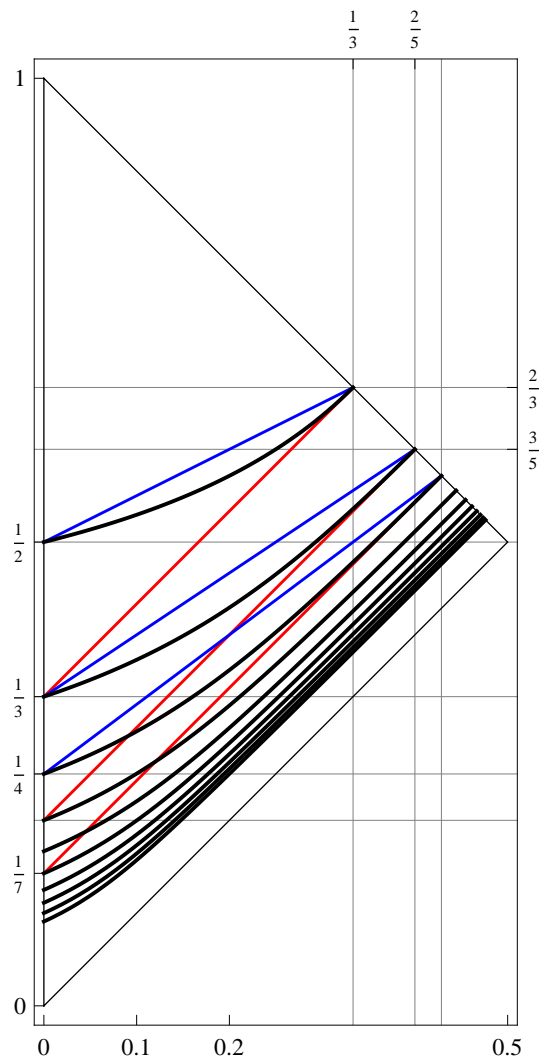


Figure 3: The black curves are where $J_n(q, p)$ vanishes. The blue lines are M_n and the red lines are L_{2n+1} , for $n = 1, 2, 3$.

8.2 The function $p_n(q)$

Our first task will be to define, and to verify the correctness of the definition, of a function $p_n(q)$ which for each $q \in (0, n/(2n+1)]$, is the unique value of p for which $J_n(q, p) = 0$. We will therefore start by proving existence and uniqueness of such a p .

Lemma 4. For $q \in T_n$, $J_n(q, 1-q) < 0$.

Proof. As in eq. (21) we have

$$J_n(q, 1-q) = \left(\frac{q}{1-q} - \frac{n}{n+1} \right) \binom{2n}{n},$$

which is negative iff $q < n/(2n+1)$. \square

Lemma 5. For $q \in T_n$, $J_n(q, q) > 0$.

Proof. The condition that $J_n(q, q) > 0$ is the same as $r_{n+1} < \rho = 1/(1-2q)$, so we must prove that $r_{n+1} < 1/(1-2q)$ when $q \in T_n$. We will prove more, namely that $r_n < 1/(1-2q)$ whenever $0 < q < 1/2$.

Let r be the fixed point that is > 1 of the Legendre polynomial recurrence, i.e. the root of the quadratic equation

$$r = \frac{(2n+1)u}{n+1} - \frac{n}{(n+1)r}$$

that is > 1 . This root is

$$W_n = \frac{\sqrt{(2n+1)^2(u)^2 - 4n(n+1)} + u(2n+1)}{2(n+1)}.$$

It is easy to check that $1 < W_n \leq 1/(1-2q)$, and tedious, but routine, to check that $W_n < W_{n+1}$ for $0 < q < 1/2$.

Now we can prove by induction that $r_n < W_n$ whenever $0 < q < 1/2$, which suffices. (Note the change from $q \in T_n$ to $q \in (0, 1/2)$; this is essential to allow the induction to carry through.) The base case can be taken to be

$$W_1 - r_1 = \frac{1}{8} \left(2q + 2\sqrt{-9(1-q)q + \frac{9}{4(1-2q)^2} - \frac{5}{4} - \frac{1}{1-2q} - 1} \right)$$

whose positivity follows from the fact that the expression is positively infinite at $q = 1/2$ and 0 only when $q = 0$.

For the inductive step, take the recurrence

$$r_{n+1} = \frac{(2n+1)u}{n+1} - \frac{n}{(n+1)r_n}$$

and assume inductively that $r_n < W_n$. Then

$$r_{n+1} < \frac{(2n+1)u}{n+1} - \frac{n}{(n+1)W_n},$$

but this last equals W_n , a fixed point of the recurrence. Therefore $r_{n+1} < W_n$, and the proof of Lemma 5 is complete because $W_n < W_{n+1}$ for $0 < q < 1/2$. \square

Theorem 8. *Given $n \geq 1$ and $0 < q \leq n/(2n+1)$. If $q \leq p < p_n(q)$, then $J_n(q, p) < 0$; if $p_n(q) < p \leq 1 - q$, then $J_n(q, p) > 0$. Thus there is only one value p for which $J_n(q, p)$ vanishes.*

Proof. This follows from Lemmas 4, 5 and 9, which tell us that $J_n(q, p)$ is a strictly decreasing function of p , going from a positive value to a negative one. \square

The existence theorem and the fact that $J_n(q, 1 - n/(2n+1)) = 0$ guarantee that $p_n(q)$ exists when $q \leq n/(2n+1)$. We now turn to the important proof that, in all cases, there is only one value p so that $J_n(q, p) = 0$.

Theorem 9. *(Uniqueness Theorem) Given $n \geq 1$ and $0 < q \leq n/(2n+1)$. If $q \leq p < p_n(q)$, then $J_n(q, p) < 0$. Thus there is only one vanishing value for J_n .*

Proof. This follows from Lemma 9 which implies that $J_n(q, p)$ is a strictly decreasing function of q ; thus it cannot return to 0 after it has once taken that value. \square

8.3 Properties of J

Now we have the functions p_n defined on T_n , with the property that $J_n(q, p_n(q)) = 0$. We will call the graph of $p_n(q)$ a *J-nullcline*, and denote it simply by p_n , or often just p . We need several properties of p_n . Note that most of the properties below have one-line proofs thanks to the efficient definition of p_n and the uniqueness result (Theorem 9).

Lemma 6. *The function p_n is continuously differentiable (C^1) on T_n .*

Proof. This is a consequence of the fact that $J_n(q, p)$ is differentiable (it is a rational function) and Lemma 5, which states that $\frac{\partial}{\partial p} J_n > 0$. These facts show that the hypotheses of the implicit function theorem are satisfied.

The preceding result is about the triangle T . But it also works on the $p + q = 1$ line, for on that line

$$J_n(q, p) = J_n(q, 1 - q) = \left(\frac{q}{1 - q} - \frac{n}{n + 1} \right) \binom{2n}{n} = \left(\frac{q}{p} - \frac{n}{n + 1} \right) \binom{2n}{n},$$

and the partial derivative $\frac{\partial}{\partial p} J_n(q, p)$ is just $-q \binom{2n}{n} / p^2$, which is nonzero. By symmetry the same proof works on the opposite side of the $p + q = 1$ line. \square

Lemma 7. (*Derivative formula*) For any point $(q, p) \in T$ and on the graph of p_n , we have

$$p'_n(q) = \frac{p(1 - p)(np - nq + p - 1)}{q(1 - q)(n(q - p) + q)}.$$

Proof. By the implicit function theorem,

$$p'_n = - \frac{\frac{\partial}{\partial q} J_n(q, p)}{\frac{\partial}{\partial p} J_n(q, p)}. \quad (23)$$

Taking the derivatives, using the recurrence to eliminate P_{n+2} , and using the $J_n(q, p) = 0$ relation to eliminate P_{n+1} in favor of P_n leads immediately to the formula. \square

Lemma 8. (*Linear vanishing condition*) $p'_n(q) = 0$ iff p_n lies on the line M_n .

Proof: Immediate from the numerator of the derivative formula (Lemma 7). \square

Lemma 9. We have

$$\frac{\partial}{\partial p} J_n(q, p) < 0 \text{ for } q \leq p \leq 1 - q.$$

Proof. The derivative inequality, after multiplication by

$$\frac{2}{q}(1 - p)^3 p(1 - q)^2 \left(\frac{-p - q + 1}{(1 - p)(1 - q)} \right)^{1-n},$$

becomes

$$P_n(u) \left(n(2p^2 - p(2q + 1) + q - 1) - 2pq + p + q - 1 \right) + (n + 1)(-p - q + 1)P_{n+1}(u) < 0,$$

and so holds precisely when $r_{n+1} < V$, where

$$V = \frac{n(2p^2 - p(2q + 1) + q - 1) - 2pq + p + q - 1}{(n + 1)(p + q - 1)}.$$

(The inequality was reversed because $-(n + 1)(1 - p - q) < 0$.)

Now $r_{n+1} < V$ can be proved by an easy induction, since the domain of truth does not depend on n . For the base case examine $V - r_1$, which works out to be the positive quantity

$$\frac{2np(1 - p)}{(n + 1)(1 - p - q)}.$$

The induction step uses the usual recurrence. Suppose $r_n < V$; then

$$r_{n+1} = \frac{(2n + 1)u}{n + 1} - \frac{n}{(n + 1)r_n} < \frac{(2n + 1)u}{n + 1} - \frac{n}{(n + 1)V}.$$

But this last is less than V because the difference $V - ((2n + 1)u)/(n + 1) - n/((n + 1)V)$ works out to

$$\frac{2n(p - q) - 2q + 1}{n(-2p^2 + 2pq + p - q + 1) + 2pq - p - q + 1}$$

in which all the grouped terms are positive. \square

Lemma 10. *Given $n \geq 1$ and $q < p$, if $J_n(q, p) = 0$ then the partial derivative $(\partial/\partial q)J_n(q, p)$ is not zero.*

Proof. We work first in T . The partial derivative of $J_n(q, p)$ can be taken and simplified using the standard recurrence for P_{n+2} , and also the relationship derived from $J_n = 0$, to replace P_{n+1} by ρP_n . This leads to

$$\frac{\partial}{\partial q} J_n(q, p) = -\frac{P_n(u)(n(p - q) + p - 1) \left(\frac{-p - q + 1}{(p - 1)(q - 1)} \right)^n}{(1 - q)^2(-p - q + 1)}.$$

The denominator is nonzero and $xy \geq 1$ in T so $P_n(u) \geq 1$; therefore the partial derivative vanishes at a point on a J -nullcline iff $n(p - q) + p - 1 = 0$ iff $p = \frac{1}{n+1} + \frac{n}{n+1}q = M_n(q)$. So we must show that this value of p cannot lead to a point at which J_n vanishes. Define $h(q) = J_n(q, M_n(q))$. This evaluates to

$$\frac{\left(\frac{-2nq + n - q}{n(q - 1)^2} \right)^n ((n + q)P_n(u) + (n(2q - 1) + q)P_{n+1}(u))}{2n(1 - q)^2}.$$

When $q = 0$, we have $u = 1$ and so $P_n(u) = 1$ and $h(0) = 0$. We also have (limits are from the left)

$$\lim_{q \rightarrow n/(2n+1)} h(q) = \lim_{q \rightarrow n/(2n+1)} J_n(q, M_n(q)) = J_n\left(\frac{n}{2n+1}, \frac{n+1}{2n+1}\right),$$

by continuity of J_n . But this last vanishes, by the diagonal formula of Theorem 5. Thus

$$\lim_{q \rightarrow n/(2n+1)} h(q) = 0.$$

Hence $h(q)$ is a differentiable function of q which vanishes at both ends of the interval $(0, n/(2n+1))$. Figure 4 shows the graphs of h for $n = 1, 2, 3, 4, 5$. It remains to show that h cannot vanish for any $q \in T_n$.

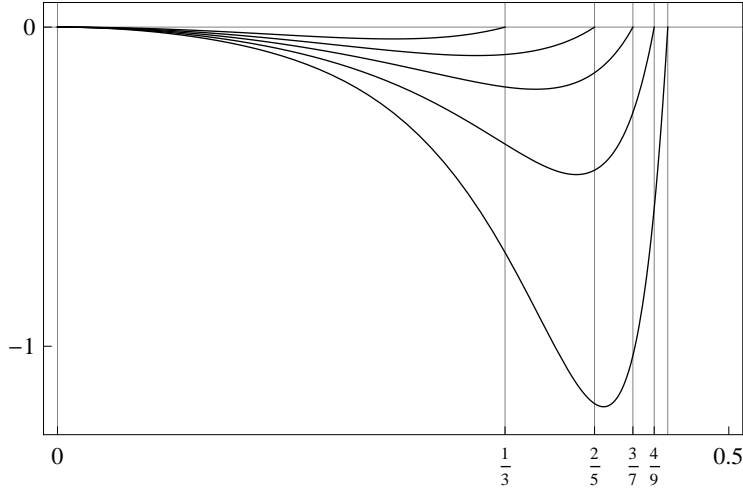


Figure 4: The graphs of $J_n(q, M_n(q))$ for $n = 1, 2, 3, 4, 5$.

Suppose $h(q)$ vanishes at some $q \in T_n$. Then, because of the vanishing at the endpoints, there must be a point q_1 such that $h'(q_1) = 0$ and $h(q_1) \geq 0$. More precisely, if there is a true crossing at q then q_1 would be given by the Mean Value Theorem applied to either $[0, q]$ or $[q, n/(2n+1)]$; if there is a tangency to the axis at q (or even if the function is identically 0), then $q_1 = q$.

So at the point $(q_1, M_n(q_1))$ (which we denote by just $(q, M_n(q))$ in the expressions below) we would have two relations for r_{n+1} : The $h(q_1) = J_n(q_1, M_n(q_1)) \geq 0$ condition means that

$$r_{n+1} \leq \rho = (1 - p + q_1)/(1 - p - q_1),$$

where in the last fraction p is to be replaced by $M_n(q_1)$; and taking the q -derivative of $h(q) = J_n(q, M_n(q))$ and recursively removing P_{n+2} leads to the following equation:

$$-(n+1) \left(\frac{n-q-2nq}{n(q-1)^2} \right)^n \frac{c_1 P_n(u) + c_2 P_{n+1}(u)}{2n(q-1)^3(nq+1)(n(2q-1)+q)} = 0,$$

where

$$\begin{aligned} c_1 &= 2n^3q + n^2(4q^3 - 4q^2 + 5q + 1) + n(2q^3 + q^2 + 2q + 1) + q(q+1), \\ c_2 &= (n+1)(1+q+2nq)(-n+q+2nq). \end{aligned}$$

Clearing the nonzero factors tells us that $r_{n+1} = -c_1/c_2$, and therefore

$$-\frac{c_1}{c_2} \leq \frac{-\frac{nq}{n+1} - \frac{1}{n+1} + q + 1}{-\frac{nq}{n+1} - \frac{1}{n+1} - q + 1}$$

which reduces to

$$\frac{2n(1-q)q}{(n+1)(2nq+q+1)} \leq 0,$$

a clear contradiction, which establishes the theorem for the triangle T . But the same proof works on the $p+q=1$ line, for on that line

$$J_n(q, p) = J_n(q, 1-q) = \left(\frac{q}{1-q} - \frac{n}{n+1} \right) \binom{2n}{n} = \left(\frac{q}{p} - \frac{n}{n+1} \right) \binom{2n}{n},$$

and the partial derivative $(\partial/\partial q)J_n(q, p)$ is just $\binom{2n}{n}/p$ which is nonzero. By symmetry the same proof works on the opposite side of the $p+q=1$ line. \square

Lemma 11. *For $q \in T_n$ we have $p'_n(q) \neq 0$.*

Proof: Because of (23), the claim follows from Lemma 10, which shows that the numerator does not vanish on the graph of p_n . \square

Lemma 12.

$$p_n \left(\frac{n}{2n+1} \right) = 1 - \frac{n}{2n+1}.$$

Proof: Follows from the diagonal equation of Lemma 2. \square

Lemma 13.

$$p'_n \left(\frac{n}{2n+1} \right) = 1.$$

Proof: The definition of p_n can be carried over by symmetry to the other side of the $p+q = 1$ line to yield a differentiable function. The implicit function theorem applies on the line itself as noted in the proof. But then, by symmetry of all the probabilities, and therefore of the vanishing of J_n , J_n is symmetric across the line. Thus differentiability implies that the derivative must be 1 to avoid a cusp. \square

Theorem 10. (*Upper bound on p_n*) For every $q \in T_n$, we have $p_n(q) < M_n(q)$, where $M_n(q)$ is the line (7) above.

Proof: Because p_n and M_n agree on the line $p+q = 1$ (Lemma 12), and because

$$p'_n \left(\frac{n}{2n+1} \right) = 1,$$

(Lemma 13) while the slope of M_n is $n/(n+1)$, the fact that p_n is C^1 (Lemma 6) means that p_n is under M_n when q is just left of the $p+q = 1$ line; because p'_n is C^1 (Lemma 6) is never 0 (Lemma 11), p_n can never cross the line M_n . \square

Lemma 14. $N(q, p_n(q)) = n$.

Proof. When $(q, p) \in T$, this follows from Theorem 2 because $J_n(q, p_n(q)) = 0$ and so $J_{n-1}(q, p_n(q)) > 0$. Therefore n is the least m such that $J_m(q, p_n(q)) = 0$. If $p_n(q) = 1 - q$ then it must be that $q = n/(2n+1)$ and, by the proof of Theorem 2, $n = q/(1-2q)$ is the unique integer such that $J_n(q, 1-q) = 0$, establishing the result. \square

Lemma 15. For every q we have $p_1(q) > p_2(q) > p_3(q) > \dots$

Proof. The graphs p_n can never cross because if $p_n(q) = p_m(q)$, $N(q, p)$ would be simultaneously n and m by Lemma 14. and the right end of p_n is above the right end of p_{n+1} (Lemma 12). So continuity of the graphs yields the result. \square

Lemma 16. The graphs of $p_n(q)$ determine the value of $N(q, p)$ exactly as follows: $N(q, p)$ is the least n such that $p \geq p_n(q)$.

Proof. We know this is correct when we are on the graph p_n (Lemma 14). But if (q, p) is between p_{n-1} and p_n then $J_{n-1} > 0$ and $J_n < 0$, by Theorem 8, and this means $N(q, p) = n$. \square

Figure 5 shows how the graphs of p_n divide triangle T into regions that define the optimal N -values, and how p_n is bounded by the two lines M_n and L_{2n+1} .

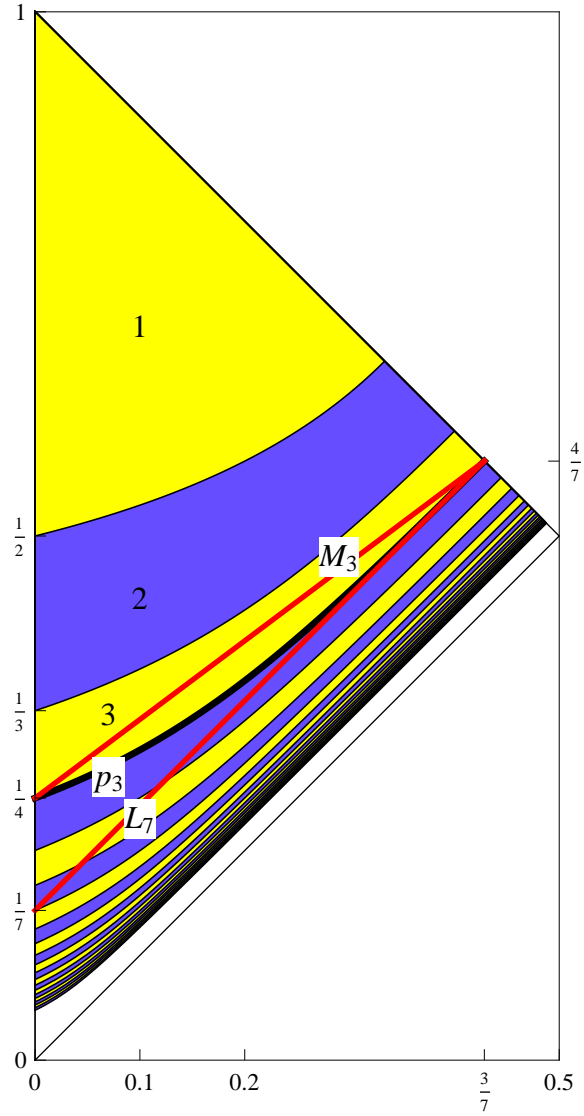


Figure 5: The graphs of p_n separate T into regions where $N = 1, 2, 3, \dots$. The two red lines M_3 and L_7 form bounds on p_3 , and this relationship underlies the lower and upper bounds on $N(q, p)$.

Corollary 2. $N(q, p)$ is a nonincreasing function of p . Thus if the p -coin becomes stronger, then the optimal choice of game length for the holder of the q -coin cannot get larger.

Theorem 11. (Upper bound for $N(q, p)$) We have

$$N(q, p) \leq \left\lceil \frac{\max(1-p, q)}{p-q} \right\rceil.$$

Proof. Assume first that $(q, p) \in T$, in which case the claimed bound is just $\lceil (1-p)/(p-q) \rceil$. Let n be the smallest value so that p lies at or above $M_n(q)$. Then $n = \lceil (1-p)/(p-q) \rceil$ and for this to be a bound we need, by Lemma 16, that $p_n(q) < M_n(q)$. But this is exactly what Theorem 10 tells us. For the special case on the diagonal line: $N(q, 1-q) = \lceil q/(1-2q) \rceil$, by the diagonal formula. But this is identical to $\lceil (1-p)/(p-q) \rceil$, which therefore works in both cases. The case in which $p+q > 1$ is handled by symmetry (see Corollary 1), with q taking the role of $1-p$. \square

The two proved bounds in Theorem 3 are equal 47% of the time because they must agree if it happens that $M_n(q) < p < L_{2n-1}(q)$; these conditions define a collection of triangles whose area, in proportion to T , is easily computed to be $\pi^2/4 - 2$. In all such cases, then, $N(q, p)$ equals $\lceil (1-p)/(p-q) \rceil$.

9 Deeper analysis of the nullclines

In section 8 we proved many properties of p_n that were evident from the graphs. We continue that here, gaining information that leads to improved bounds on N and to an efficient algorithm for computing N (section 10). We first observe that when n is small, p_n is given by simple formulas. Such formulas are useful as a check on computations.

Lemma 17.

$$\begin{aligned} p_1(q) &= \frac{1-q}{2-3q}, \\ p_2(q) &= \frac{1-q}{3-12q+10q^2} (2-4q-\sqrt{1-4q+6q^2}). \end{aligned}$$

Proof. Just solve the polynomial equation $J_n(q, p) = 0$. There are more complicated radical expressions for p_2 , p_3 , and p_4 . The last is ostensibly a quintic, but the polynomial in the equation is divisible by $1-p$ yielding a quartic equation. \square

Because J_n is infinitely differentiable in T , the fact that the hypothesis of the implicit function theorem is met (Lemmas 9 and 6) means that p_n is infinitely differentiable. The second derivative is easily computed by implicitly differentiating the derivative formula (Lemma 7).

Lemma 18. *We have*

$$p_n''(q) = \frac{p(1-p)(1-p-q)}{q^2(q-1)^2(n(p-q)-q)^3}Z,$$

where Z is given by

$$\begin{aligned} & 2n^3(p-q)^3 + 2n^2(2p^3 - p^2(7q+1) + pq(7q+3) - 2q^2(q+1)) + n(2p^3 - p^2(11q+2) \\ & + pq(11q+10) - q(2q^2+7q+1)) - (p-1)q(3p-3q-1). \end{aligned}$$

But we can prove the weaker and still very useful assertion that the slope never rises beyond its value at the right end.

Lemma 19. *For $q \in T_n$ we have $0 < p_n'(q) < 1$.*

Proof. The lower bound holds because the slope is 1 at the right end (Lemma 13) but never vanishes (Lemma 11), and so must stay positive. For the upper bound, if $p_n'(q) \geq 1$ then, because the slope is 1 at the right end, the Mean Value Theorem gives q_1 with $p'(q_1) \geq 1$ and $p''(q_1) = 0$. But when we insert the formulas for the derivatives (Lemmas 7 and 18) into these two relations we get a contradiction. This is most easily done by Mathematica's **Reduce** function, which gives **False** in an eyeblink. Working out the contradiction by hand is possible, but tedious. \square

While the original probability formulation makes no sense when $q = 0$ (there is no optimal choice of N when the underdog loses each play), the limit $\lim_{q \rightarrow 0} p_n(q)$, evident from Figure 3, is quite simple.

Lemma 20. $\lim_{q \rightarrow 0^+} p_n(q) = \frac{1}{n+1}$

Proof. Because $p_n'(q) > 0$ (Lemma 19), the values $p_n(q)$ decrease as $q \rightarrow 0$; the values are bounded and so the claimed limit exists. Thus we will use $p(0)$ to denote $\lim_{q \rightarrow 0^+} p_n(q)$. Further, $0 < p(0) < 1$, for by Lemma 3, $p_n(q) > 1/(2n+1) + q$ so $p(0) \geq 1/(2n+1)$; and $p(0) < 1 - n/(2n+1)$ because the derivative is positive and $p_n(n/(2n+1)) < 1$ (Lemma 12). Now, the derivative formula tells us that

$$\frac{((p-1)p(np-nq+p-1))}{((q-1)q(n(q-p)+q))} - p'(q) = 0,$$

where p denotes p_n . Multiplying both sides by the denominator turns this into

$$(p-1)p(np-nq+p-1) - (q-1)q(n(q-p)+q)p'(q) = 0.$$

But the boundedness of p'_n means that the limit of $qp'(q)$ is 0, giving

$$\lim_{q \rightarrow 0^+} p(p-1)(np+p-1) = 0,$$

or $(p(0)-1)p(0)(p(0)(n+1)-1) = 0$. Only the last factor can vanish, giving $p(0) = 1/(n+1)$. \square

We next move to a proof of the (computationally evident) fact that the J -nullclines are convex. Figure 6 shows the second derivatives of p_n for $n \leq 15$. They are evidently unbounded at $q = 0$, converging to 0 at the right, and always positive. We will now prove positivity (i.e., convexity of the J -nullcline curves), which will be important as a source of new approximations to $N(q, p)$.

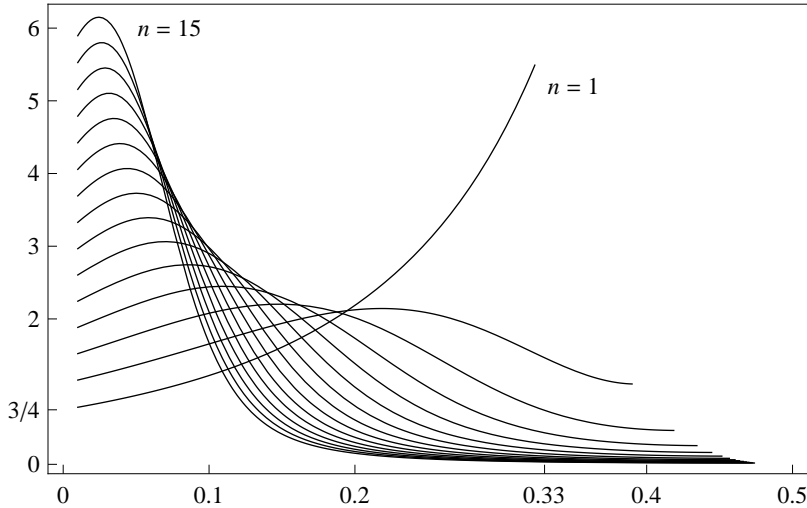


Figure 6: The graphs of $p''_n(q)$ for $n \leq 15$.

Theorem 12. (*Convexity*) $p''_n(q) > 0$ in T_n .

Proof. Because the first derivative of p_n is 1 at the right end of T_n (Lemma 13) and less than 1 elsewhere (Lemma 19), the second derivative is positive somewhere and it is continuous

by the implicit function theorem (comment after Lemma 17). Therefore it suffices to prove that $p_n''(q)$ never vanishes. The second derivative is given by the formula of Lemma 18. We can eliminate nonzero factors, thus reducing its vanishing to the following equation in p .

$$\begin{aligned} & (2n^3 + 4n^2 + 2n) p^3 + p^2 (-6n^3 q - 2n^2(7q + 1) - n(11q + 2) - 3q) \\ & + p (6n^3 q^2 + 2n^2 q(7q + 3) + nq(11q + 10) + 3q^2 + 4q) - nq (2q^2 + 7q + 1) \\ & - 3q^2 - q - 2n^3 q^3 - 4n^2 q^2 (q + 1) = 0 \end{aligned}$$

Fixing n and q , the vanishing condition is a cubic in p . The cubic always has a real root and checking the critical points assuming $0 < q < 1/2$ and $n \geq 1$ shows that they never straddle 0, which means that the other two roots are never real. Call the unique real root $p_n^-(q)$. The theorem will be proved once we show that $J_n(q, p_n^-(q))$ in T_n , so that the inflection point is not on the graph of p_n . We can get a closed form for $p_n^-(q)$ by solving the cubic, but we do not need the explicit representation as the needed algebra can be worked out implicitly from the cubic relation. Yet it is instructive to look at $p_n^-(q)$. Figure 7 shows its graph for $n \leq 5$, together with the graphs of p_n in pink, and also the base-10 log of the difference for $n = 1, 2, 3, 10, 20, 100$. The inflection curve is barely below the J -nullcline and the two curves are visually indistinguishable.

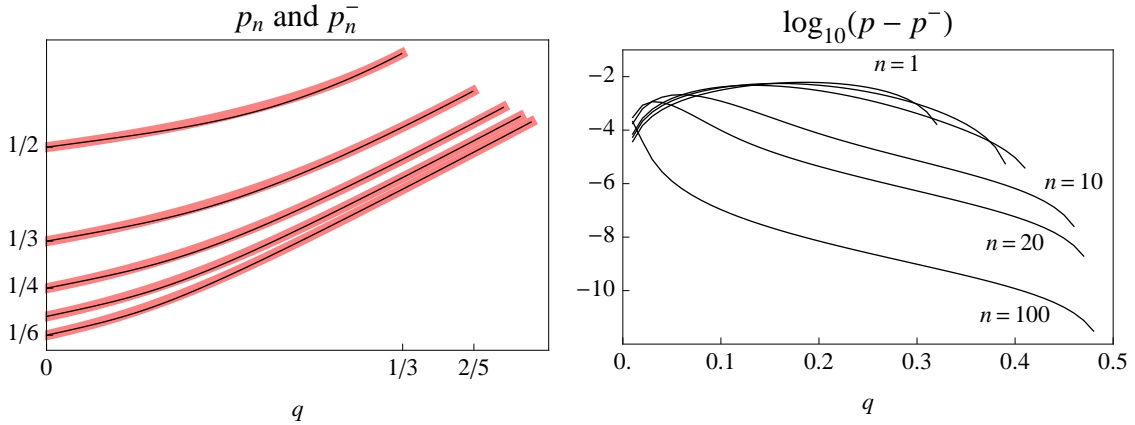


Figure 7: Left: The graphs of p_n (pink) and $p_n^-(q)$ (black). Right: The difference in the graphs viewed through a logarithmic lens.

We can use implicit differentiation on the defining cubic to get a formula for $\frac{d}{dq}p_n^-(q)$. Since $p_n^-(q)$ is given explicitly by radicals we know its derivative exists. The implicit derivative formula $p_n^-(q) = -\partial_q/\partial_p$ then gives the following representation for the derivative $\frac{d}{dq}p_n^-(q)$:

$$\frac{(6n^2 + 8n + 3)p^2 - 2p(6n^2q + n(8q + 3) + 3q + 2) + 6n(n + 1)q^2 + (8n + 6)q + 1}{6n^2(p - q)^2 + 2n(3p^2 - 2p(4q + 1) + q(4q + 3)) + q(-6p + 3q + 4)}. \quad (24)$$

Now we follow the proof idea of Lemma 10. We need to show that, given n , the point $(q, p_n^-(q))$ cannot lie on the J_n -nullcline. Define the function $h(q) = J_n(q, p_n^-(q))$, which we claim does not vanish in T_n . Figure 8 shows the first few graphs of h .

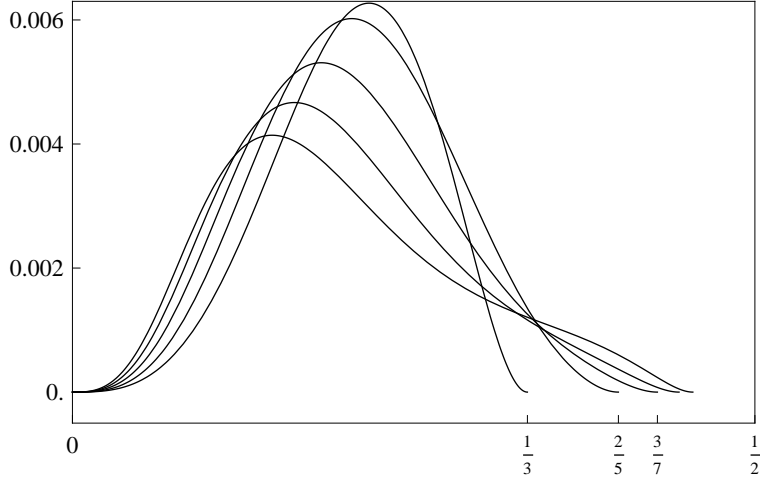


Figure 8: The graphs of $J_n(q, p_n^-(q))$ for $n \leq 5$.

We observe that h vanishes at both ends. At the left this is because $J_n(0, p)$ is always 0 (the Legendre terms become just 1). At the right we find that the defining cubic, when q is set to $n/(2n+1)$, has the factor $2np - n + p - 1$, which means that $p = 1 - n/(2n+1)$ is a root. The cubic has exactly one real root in the domain of interest, so $1 - n/(2n+1) = p_n^-(\frac{n}{2n+1})$ and we know (Lemma 2) that $J_n((n/(2n+1), 1 - n/(2n+1))) = 0$.

Now suppose the graph is 0 at some $q_1 \in T_n$. Then looking left and right of q_1 and using the Mean Value Theorem, we get a value, call it q , such that $h(q) \leq 0$ and $h'(q) = 0$. Recall that $h(q) \leq 0$ means that $r_{n+1} = P_{n+1}/P_n \geq \rho = (1-p+q)/(1-p-q)$, where p denotes $p_n^-(q)$. When we form $h'(q)$ leaving $p_n^-(q)$ undefined and then substitute the derivative formula (24) for the derivatives of $p_n^-(q)$ that appear, we obtain the familiar form $c_1P_n + c_2P_{n+1} = 0$. This

becomes $-c_1/c_2 = r_{n+1}$, so we have the relation $-c_1/c_2 \geq \rho$. If we work out c_1, c_2 in terms of n, p, q we get a rational function of these three variables which is too large to reproduce here, but which can be found at [5]. A call to Mathematica's `Reduce` function shows, in two seconds, that the five conditions:

1. $-c_1/c_2 \geq \rho$,
2. the vanishing of the second derivative formula at p ,
3. $0 < q < p < 1 - q$,
4. $n \geq 1$, and
5. $q < n/(2(n + 1))$

are contradictory. For such polynomial systems `Reduce` uses a cylindrical algebra decomposition [3]; this example requires showing that there is no solution in each of 1062 cylindrical cells. Working this out by hand might be extremely difficult, if not impossible. \square

Corollary 3. (*Slope convergence*) *The limit $\lim_{q \rightarrow 0^+} p'_n(q)$ exists.*

Proof. The slopes are bounded (Lemma 19) and monotonic by the convexity of p_n . \square

Corollary 4. (*Slope-limit formula*) $\lim_{q \rightarrow 0^+} p'_n(q) = n/(2(n + 1))$.

Proof. Let S denote the limit. Because the derivative formula $p' = ((1 - p)p(np - nq + p - 1))/((1 - q)q(nq - np + q))$ holds for the slopes, we want the limit of this expression as $q \rightarrow 0$. Now, as $q \rightarrow 0$, $p \rightarrow 1/(n + 1)$. So we can look at the numerator and denominator separately and see that we have a $0/0$ form, and we can use l'Hopital's rule to get the limit, where we use $p(q)$ for p . Forming the l'Hopital quotient, using $\lim p(q) = 1/(n + 1)$, $\lim p'(q) = S$, and then letting q be 0 yields $(n - (n - 1)S)/(n + 1)$. Setting this to equal to S and solving gives the formula. \square

Corollary 4 allows us to think of p_n as a C^1 function on all of \mathbb{R} as follows. Let S_n be the limit of the slopes that the corollary provides. Define p_n^* to agree with p_n on $[0, n/(2n + 1))$ and to be the linear function through $(0, p_n(0))$ of slope S_n on $(-\infty, 0]$, and the similar tangent-line extension on the right. It is easy to see using the Mean Value Theorem and the limit definition of S_n that the limit of the slopes of the secants connecting $(0, p_n(0))$ to $(q, p_n(q))$ is S_n , giving the continuous differentiability of p_n^* .

Corollary 5. (*Slope bound*) $p'_n(q) > \frac{n}{2(n+1)}$ in T_n .

Proof. By the previous corollary, because the slopes are monotonically increasing, by convexity. \square

Corollary 6. (*Linear lower bound*) In T_n , $p_n > K_n(q) \stackrel{\text{def}}{=} \frac{1}{n+1} + \frac{n}{2(n+1)}q$, and

$$N(q, p) \geq \lceil (1-p)/(p-q/2) \rceil$$

Proof. The linear function K_n agrees with p_n at $q = 0$ by Lemma 20. If p_n dipped below K_n , the Mean Value Theorem would provide a point contradicting Corollary 5. Inverting the bound on p gives a bound on N . \square

The upper bound on N given by Theorem 11 and the piecewise lower bound obtained by combining Theorem 6 and Corollary 6 are useful computationally. For when the two bounds agree we know immediately that $N(q, p) = \lfloor 1/(2(p-q)) + 1/2 \rfloor$. And in cases where N equals the piecewise lower bound, that can be verified by a single J -evaluation: just check that $J_n(q, p)$ is negative when n is the lower bound. The subset of T for which the two bounds agree is a union of triangles – the green region in Figure 9 – and the total area of these triangles can be determined by integrating Mathematica’s `Boole` function to get an expression for each level and then summing the results symbolically. The total area of the region in which N equals the lower bound can be approximated by experimentation using thousands of points. The results are summarized in the next theorem.

Theorem 13. 1. For (q, p) chosen uniformly from T , the probability that the upper bound of Theorem 11 and the combined lower bounds of Theorem 6 and Corollary 7 agree is

$$(1-i)\psi(3-i) + (1+i)\psi(3+i) + \frac{\pi^2}{4} + 2\gamma - \frac{115}{27},$$

where ψ is the digamma function Γ'/Γ , and γ is Euler’s constant. This is roughly 0.60.

2. For (q, p) as above, the probability that

$$N(q, p) = \max \left(\left\lfloor \frac{1-p}{p-\frac{q}{2}} \right\rfloor, \left\lfloor \frac{1}{2(p-q)} - \frac{1}{2} \right\rfloor \right) \quad (25)$$

is approximately 0.87.

Corollary 7. (*Improved bounds on N*) In T , $N^-(q, p) \leq N(q, p) \leq N^+(q, p)$, where N^- and N^+ are, respectively, the positive-radical solutions, n , to the quadratic equations:

$$2n^2(p-q)^3 + 2n(p^3 - p^2(4q+1) + pq(4q+3) - q^2(q+2) - (p-1)q(3(p-q)-1)) = 0,$$

and the equation $DF = n/(2(n+1))$, where DF is the derivative formula of Lemma 7.

Proof. For the lower bound, the equation is equivalent to setting the second derivative formula to 0 and clearing nonzero terms. One first observes that, fixing q and p , the second derivative can only rise with n . This follows from an application of **Reduce** as usual, showing that the negativity of ∂_n of the second derivative formula is inconsistent with (a) membership in T , and (b) $K_n(q) < p$, and (c) $L_{2n+1}(q) < p$. Because $N(q, p)$ is not less than the least n such that $p_n''(q) = 0$, the result follows. The upper bound is obtained the same way, using the fact that the slope is not less than $n/(2(n+1))$, which becomes a quadratic relation. \square

Expanding the rational expression for N^- in a series in powers of $p - q$ shows that it equals $1/(2(p - q)) - 3/2 + 1/(4p(1 - p)) + O(p - q)$, which relates it nicely to the simpler bound of Theorem 6. Define $H(q, p) = \lceil 1/(2(p - q)) - 3/2 + 1/(4p(1 - p)) \rceil$; while H is not a lower bound on N , it is a useful approximation when p is close to q and appears to be asymptotically perfect when the domain is rescaled (see subsection 9.1).

One can view the quadratic equations that give N^\pm as cubic equations in p , in which case solving gives radical expressions for p_n^\pm , which bracket the curve p_n . These expressions are useful when generating images such as Figure 9.

We can measure how good the improved bounds are by assuming that the point (q, p) is uniformly distributed in T . Then one can ask: (1) How often does $N = N^+$? (2) How often does $N = N^-$? The answers are: “remarkably often.” Figure 10 shows points in T colored green if both bounds agree, red if the lower bound is correct, and yellow otherwise. The upper bound is sometimes correct, but not often. Of course, when the bounds agree we know N immediately, and when the lower bound is correct, then that can be proved by verifying that $J_{N^-}(q, p) < 0$ (more on computing J in section 10 below). The blue curves are the graphs of p_1^\pm ; they bracket p_1 , defined by the yellow-red boundary. In Figure 10 the green area where the two bounds agree is 75% of the triangle and the green-plus-red area where N equals the lower bound is 97% of the area.

9.1 A harmonic rescaling

The views of Figures 9 and 10 do not show clearly what happens near the $p = q$ line. We can take a microscope to that area by harmonically rescaling the domain. We do this by first rotating T clockwise 45° and then stretching out the vertical scale: precisely, after the rotation we change each y -coordinate to $-(1/(2\sqrt{2}y)) - 1/2$. This is essentially just a change of coordinates from (q, p) to $(p + q, 1/(p - q))$; Figures 11 and 12 show the view through this microscope. The two approximations N^- and H exactly equal N a large percentage of the time. For the region defined by $N \leq 5000$ we found that $N = N^-$ in 99.94% of the region while N and H agree 96% of the time. Thus we can conjecture that the probability of either

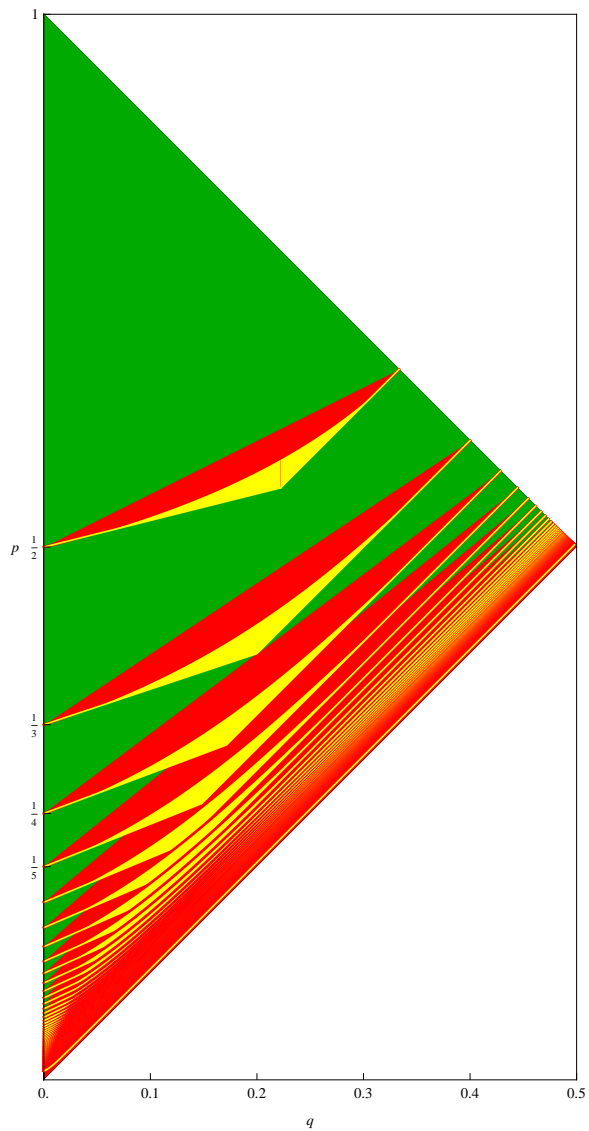


Figure 9: The green region is where the two bounds on N agree; the red region is where N agrees with the lower bound. Thus in the combined red and green regions $N(q, p)$ is expressible exactly as $\max\left(\left\lceil \frac{1-p}{p-q} \right\rceil, \left\lfloor \frac{1}{2(p-q)} - \frac{1}{2} \right\rfloor\right)$.

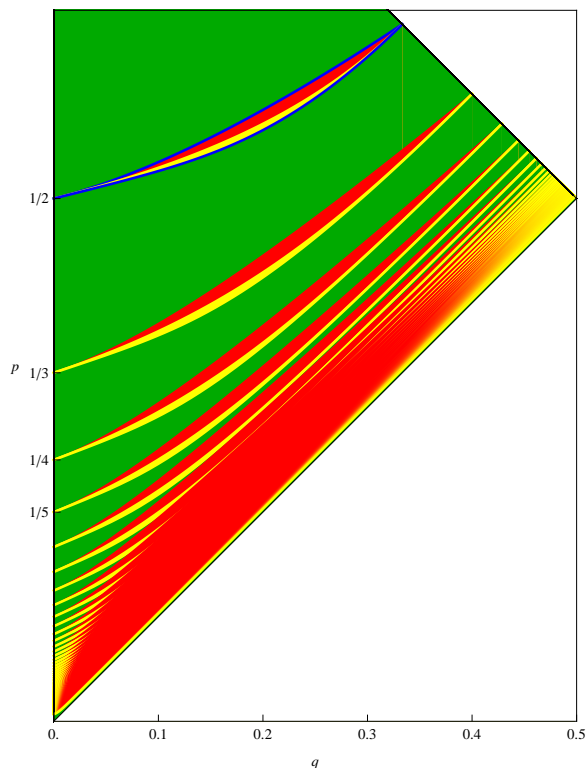


Figure 10: The green region is where $N^- = N^+$; red is where $N^- = N < N^+$, and yellow is the rest. The blue curves are the graphs of p_1^- and p_1^+ .

equation holding is asymptotically 1.

10 An algorithm for computing $N(q, p)$

A straightforward algorithm to compute $N(q, p)$ first uses symmetry to restrict to T and then finds the smallest integer n such that $p \geq p_n(q)$; that value of n is $N(q, p)$ by Lemma 16. One can start with the simple or improved bounds and then use either bisection or the secant method, repeatedly checking whether $J_n(q, p)$ is positive or negative. This method works fine when N is of modest size, but when N is large the Legendre polynomials cannot be explicitly computed. A solution is to use the integral formula given in (20), which is a fine substitute for P_n . That formula means that we can determine the sign of J_n for each

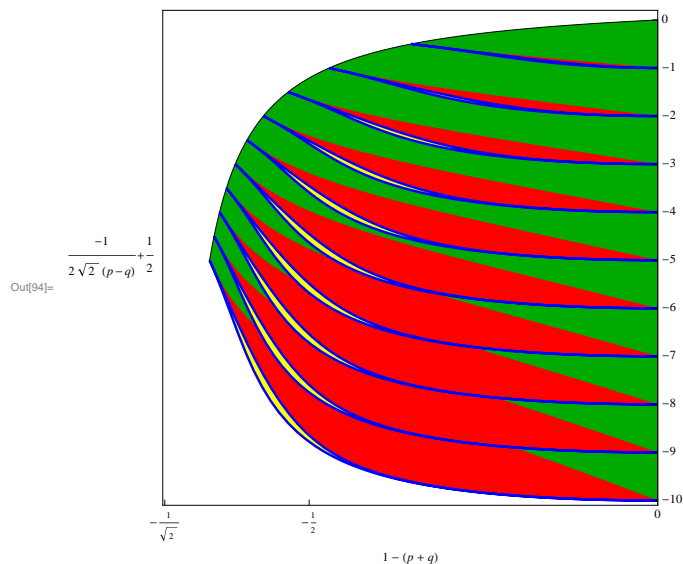


Figure 11: A rotated and vertically stretched view of the (q, p) domain. The colors have the same meanings as in Figure 10. We see here that the region (yellow) where N is not equal to N^- is very small.

trial by using numerical integration on

$$(1 - p + q) \int_0^\pi \left(u + \sqrt{u^2 - 1} \cos t\right)^n dt - (1 - p - q) \int_0^\pi \left(u + \sqrt{u^2 - 1} \cos t\right)^{n+1} dt.$$

Of course, high-precision must be used as appropriate. One needs enough accuracy to account for the full precision of n which will be used as a trial in the root-finding process. But one needs only the sign of the integral above. In Mathematica this means that when computing the integral numerically one needs a large working precision, but the accuracy goal can be quite small. The method is robust and takes only a few seconds to compute $N(10^{-100}, 2 \cdot 10^{-100})$, which is

72768 90317 94675 98852 95987 53552 38752 84521 10838 88022 00705 28794 63897 19626 49789
77512 24788 32188 39061 36928.

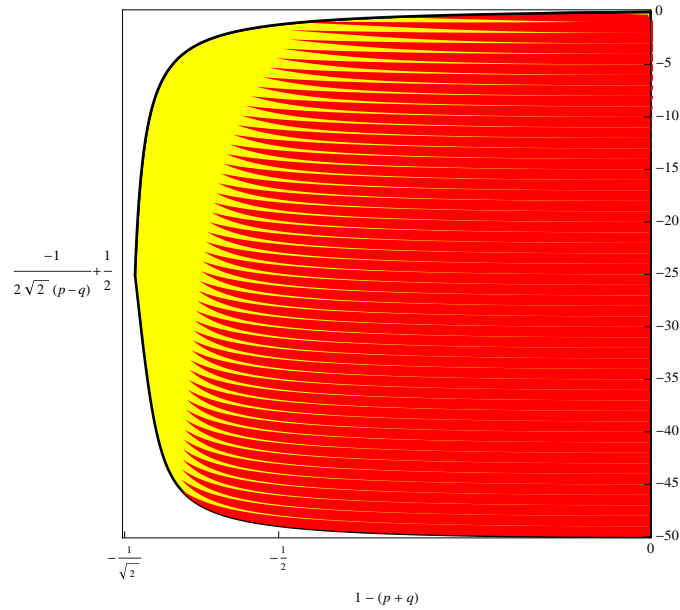


Figure 12: A rotated and vertically stretched view of the (q, p) domain in the region where $N \leq 50$. Here red is where $N(q, p) = H(q, p)$, yellow where they are not equal. The proportion of the space in which H is correct is 76%, but it appears to converge to 100% as $N \rightarrow \infty$.

q	p	$N(q, p)$
10^{-5}	$2 \cdot 10^{-5}$	72768
10^{-10}	$2 \cdot 10^{-10}$	7276890317
10^{-15}	$2 \cdot 10^{-15}$	727689031794675
10^{-20}	$2 \cdot 10^{-20}$	72768903179467598852
10^{-25}	$2 \cdot 10^{-25}$	7276890317946759885295987
10^{-30}	$2 \cdot 10^{-30}$	727689031794675988529598753552

Table 1. The integration algorithm allows one to get giant values of $N(q, p)$.

When one wants not just the sign of J_n , but a numerical approximation to the full J_n -nullcline – the graph of p_n – one can use a numerical differential equation approach. Because

of the derivative formula and the known values at 0, we can set up the initial-value problem as $p(0) = 1/(n+1)$ and

$$p'(q) = \frac{p(1-p)(np - nq + p - 1)}{q(1-q)(n(q-p) + q)}$$

if $q > 0$ and $n/(2(n+1))$ if $q = 0$. This approach is a quick way to generate graphs of p_n , such as those shown in various figures in this paper. It can also be used in an algorithm for computing $N(q, p)$ where it can sometimes be faster than the use of numerical integration because the solution is needed only on the interval $[0, q]$, while the integrals are computed from 0 to π .

11 Some open questions

1. Improve the bounds on $N(q, p)$.
2. Find a more efficient algorithm for computing $N(q, p)$ when (q, p) is near the origin.
3. Generalize, in a natural way, these results to the case of three players.
4. Find a direct, elementary proof of Theorem 1, i.e., a proof that makes no use of any computer algebra packages or algorithms.
5. Prove the second derivative conjectures, which we obtained heuristically by manipulating Taylor polynomials:

(a)

$$\lim_{q \rightarrow 0^+} p_n''(q) = \frac{2n^2 + 5n + 2}{6(n+1)},$$

and

(b)

$$\lim_{q \rightarrow (\frac{n}{2n+1})^-} p_n''(q) = \frac{4(2n+1)}{n(2n^2 + n - 1)}.$$

6. (Asymptotic closed form conjectures)
 - (a) If q and p are chosen uniformly from the harmonically rescaled domain then the probability that $N(q, p) = N^-(q, p)$ approaches 1 as $N \rightarrow \infty$.
 - (b) Same as above, but with the conclusion that $N = \lceil 1/(2(p-q)) - 3/2 + 1/(4p(1-p)) \rceil$ with asymptotic probability 1.

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Macalester College, St. Paul, MN, 55105; <addona@macalester.edu>

Macalester College, St. Paul, MN, 55105; <wagon@macalester.edu>

University of Pennsylvania, Philadelphia, PA 19104-6395; <wilf@math.upenn.edu>