AMS / MAA | PROBLEM BOOKS VOL 36

BICYCLE OR UNICYCLE?

DANIEL J. VELLEMAN | STAN WAGON

A Collection of Intriguing Mathematical Puzzles

MAA PRESS
An Imprint of the American Mathematical Society

Contents

Preface

Mathematical problems can be appealing for a variety of reasons, but we are especially fond of problems that have a big surprise factor. The surprise can take different forms. Our previous problem book, *Which Way Did the Bicycle Go?* [**82**], highlighted a problem in bicycle geometry where the surprise was a bit unusual: the fact that Sherlock Holmes made a serious error in logical reasoning. The present book starts with another bicycle problem, one for which the surprise is that a bicycle can pretend to be a unicycle.

We have selected the 105 problems in this collection with an eye to the surprise factor. Almost all of these problems appeared in the Problem of the Week program at Macalester College. After his retirement from Macalester, Stan Wagon kept the program going via a mailing list.

The surprises come in a variety of forms. For some problems the goal is to determine an integer, but at the start one would have no idea if the size of the answer is near 10, 100, or $10^{10^{100}}$. A remarkable one of this type is a cake-slicing problem (Problem 21) that everyone who guesses at the answer gets wrong. In this case, the answer is quite different from what one would expect. Three other examples are Problems 37, 38, and 91.

An important aspect of problem posing is getting the statement exactly right. A notorious probability problem (Problem 49) concerns a parent who has two children, one of whom is a son born on a Tuesday. This particular problem has generated an enormous amount of discussion, as the result appears to be paradoxical. But the key is how the problem is posed, and we have done so in a way that makes it possible to clear up the mystery.

A different type of paradox, one that highlights the difference between large finite sets and infinite sets, is discussed in Problem 79. That problem shows the importance of the choice of fundamental axioms for mathematics.

Two problems (Problems 90 and 96) show that contest problems can provide a big surprise to the writers of the contest, as the problems turned out to be much more delicate than the designers intended.

Situations where π appears unexpectedly have always fascinated mathematicians and the general public. There are the classics such as the Buffon needle problem with its probability of $2/\pi$ and the sum of the reciprocals of the square integers, which Euler proved to be $\pi^2/6$. For one problem in this collection π makes an appearance that is just as, or more, surprising than the two classical examples.

Problem 95, about the ways in which polynomials change as they move from left to right, shows that new discoveries can be made involving very elementary notions.

As is traditional in mathematics, some problems are based on ideas from physics. Trajectories of projectiles are considered in Problems 61 and 62. Problem 66 concerns the center of mass of a partially filled glass of water, and Problem 105 deals with collisions between sliding blocks. Also, again as is traditional, problems lead to further problems. Problem 81 is a not-too-difficult problem about integers. But it leads to a much more surprising real-number version, Problem 82, whose solution uses entirely different methods. And many problems here lead naturally to open questions.

Sometimes popular culture or games can lead to interesting problems. Five of that sort are related to the games of poker (Problem 54) and Bingo (Problem 56), the culture of crossword puzzles (Problems 92 and 93), and a puzzle feature of the first Harry Potter book (Problem 94).

Some of our problems have short and concise solutions, while others are quite complicated. Almost all can be done by hand, but occasionally help from computers is needed, and we view such a tool as indispensable for investigations into many problems. While almost all of our problems are typical of the genre, in that the solutions use standard mathematical tools and are not overly complicated, we have included a few that really are research projects (e.g., Problem 1). In a couple of cases, a rigorous proof requires a result that might not be well known and we have included the full details.

Acknowledgments. Many readers of the Macalester Problem of the Week have contributed valuable solutions and comments over the years. Thus we are grateful to many people for their enthusiasm and insights. In particular, we thank Larry Carter, Joseph DeVincentis, Michael Elgersma, Jim Guilford, John Guilford, Witold Jarnicki, Stephen Morris, Rob Pratt, Peter Saltzman, Richard Stong, Jim Tilley, and Piotr Zielinski. And we also thank Joe Buhler for his discussions on problems and the innovative problem section he (together with the late Elwyn Berlekamp) has written for *Emissary*, the newsletter of the Mathematical Sciences Research Institute.

Dan Velleman, Amherst College and University of Vermont
`djvelleman@amherst.edu`

Stan Wagon, Macalester College and Silverthorne, Colorado
`wagon@macalester.edu`

Notation. We use the following notation. $\mathbb{N} = \{0, 1, 2, \ldots\}$, the natural numbers; $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the integers. $[n]$ denotes $\{1, 2, \ldots, n\}$. A permutation is denoted as a row of numbers, such as $3\,1\,4\,2$ for $\binom{1\,2\,3\,4}{3\,1\,4\,2}$, which is the cycle $1 \to 3 \to 4 \to 2 \to 1$. This cycle is also denoted $(1\,3\,4\,2)$. An approximation to a real number is denoted by $\approx$, as in $\pi \approx 3.142$.

1

Can a Bicycle Simulate a Unicycle?

1 Bicycle or Unicycle?

The wheels of a bicycle generate two tracks. When the front wheel turns, the tracks are generally distinct, while for a nonturning bike, the track is a straight line and the rear wheel follows exactly in the track of the front wheel; in that case one cannot tell whether the track was made by a bicycle or a unicycle. Show that there can be a nonstraight bicycle track for which the rear wheel follows the front track exactly, and with the rear track overlapping the front track by at least several bike lengths.

We represent a track as a continuously differentiable curve or path in the plane with derivative never being the zero vector. Assume the bicycle's wheel base is 1 unit; then because the rear wheel does not steer, the unit tangent vector at any point of the rear path ends on the front path (Figure 1.1).

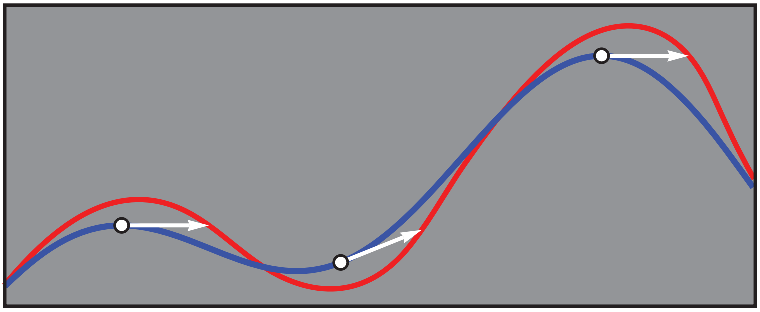


Figure 1.1. Typical tracks made by the front and rear wheels of a bicycle. The unit tangent to the blue rear track determines the front track.

5 Probability

49 The Importance of Irrelevant Information

(a) A random parent of two children is asked, “Do you have a son?” If the answer is “Yes,” what is the probability that both children are sons?

(b) A random parent of two children is asked, “Do you have a son born on a Tuesday?” If the answer is “Yes,” what is the probability that both children are sons?

Assume that the boy/girl probability is 1/2, all days of the week are equally likely for births, and the genders and birth days of the two children are independent.

50 Creeping Ants

At a certain time, 101 ants are placed on a one-meter stick. One of them, Alice, is placed at the exact center, and the positions of the other 100 ants are random, as are the directions they face. All ants start crawling in whatever direction they are facing, always traveling at one meter per minute. When an ant meets another ant or reaches the end of the stick, it immediately turns around and continues going in the other direction. What is the probability that after one minute Alice is at the exact center of the stick?

51 Roll the Dice

What is the expected number of distinct faces seen when rolling a 6-sided die n times?

52 Conditioned Throws of a Die

(a) A 3-sided die with numbers 1, 2, and 3 is thrown until a 1 appears. What is the expected number of throws (including the throw giving 1)?

(b) A standard 6-sided die is thrown until a 1 appears. What is the expected number of throws (including the 1) conditioned on the event that all throws gave values 1, 2, or 3? The condition means that any string of throws with a 4, 5, or 6 does not contribute to the expected length.

53 Where Are the Rounded Powers of Two?

Consider the positive integer powers of 2, and round each one to just one significant digit; in other words, round to the nearest number of the form $d \cdot 10^k$, where d is one of the digits 1 through 9. In case of a tie, round up. What is the most likely leading digit of the result?

More precisely, the question should be interpreted asymptotically. For each positive integer N, one can compute the proportion of the first N rounded positive powers of 2 that start with each of the nine digits; this gives the probability of each digit occurring as the leading digit if one of the first N rounded powers of 2 is chosen at random. Then take the limit as N approaches infinity. For which digit is this limiting probability the largest, and what is the limiting probability?

54 The Holy Game of Poker

You and a few friends are playing stud poker (five cards dealt, no exchanging of cards) with a standard 52-card deck. It's your lucky day and God tells you that you are guaranteed to end up with a full house, but only if you correctly choose the best full house, meaning the one with the highest probability of winning. Which full house should you choose?

The ranking of poker hands, from lowest to highest, is: high card, one pair, two pair, three of a kind, straight, flush, full house, four of a kind, straight flush. Only the last three are relevant.

- a *full house* is a pair and triplet, such as 2♠, 2♥, 5♥, 5♣, 5♠;
- *four of a kind* is a quadruplet, such as 5♠, 5♥, 5♣, 5♦, 2♥;
- a *straight flush* is five in a row of the same suit, such as 2♠, 3♠, 4♠, 5♠, 6♠. An ace can be either the high or low card of a straight flush, so both A 2 3 4 5 and 10 J Q K A, all in the same suit, count as straight flushes.

For two full houses, the one with the higher triplet wins (ace being high).

55 A Shrinking Random Walk

Consider a random walk in the plane that starts at the origin and moves only in the positive x and y directions. The direction choice at each step is governed by the flip of a fair coin but the length shrinks: the length of the nth move is 2^{-n} (so the first move has length $1/2$; the second has length $1/4$, and so on; see Figure 55.1). If L is the limiting point of the random walk, what is the expected distance of L from the origin?

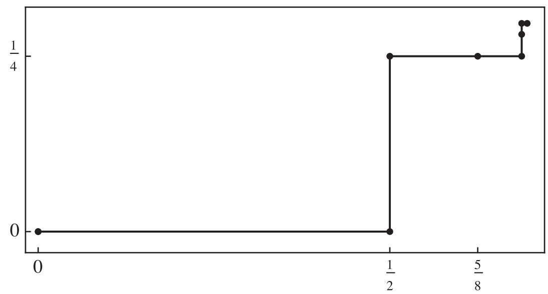


Figure 55.1. A random walk that only goes east or north, with each step half the size of the preceding step.

56 BINGO!

Consider a BINGO game in which every possible card is in play. Random calls are made from B1, B2, …, B15, I16, …, I30, N31, …, N45, G46, …, G60, O61, …, O75. What is the probability that the first win is horizontal, as opposed to vertical?

Details. BINGO is played with cards such as the one in Figure 56.1, with random calls from the 75 numbers until someone in the room claims a win. A win consists of a row or column of chosen numbers. Traditional BINGO wins include the two diagonals, but we are not considering those here. Moreover, a traditional card has a "free square" at the center, but for this problem the center is just another N-number.

57 How Much is a Penny Worth?

Alice tosses 99 fair coins and Bob tosses 100. What is the probability that Bob gets strictly more heads than Alice?

B	I	N	G	O
13	17	40	51	68
3	29	34	57	70
9	26	33	50	67
11	25	31	46	74
8	16	45	52	69

Figure 56.1. A BINGO card has 25 numbers from 1 through 75, with five chosen from each successive group of 15.

58 A Historically Interesting Truth-Teller Problem

Suppose Alice, Bob, Charlie, and Diane tell the truth with (independent) probability 1/3. Alice stated that Bob denied that Charlie declared that Diane lied. What is the probability that Diane told the truth?

Details. We assume that Diane made a certain statement whose truth value both Charlie and Diane knew (but of course Diane lied about it with probability 2/3). Then Charlie said one of "Diane told the truth" or "Diane lied" according to what he heard and his own probability of lying. Bob heard what Charlie said, and he said one of "Charlie said that Diane told the truth" or "Charlie said that Diane lied." And Alice heard Bob's statement and made her assertion accordingly.

59 Monty Hall Revisited

Recall the famous Monty Hall problem: Monty Hall is the host of a game show on which there are three doors, with a car behind one door and goats behind the other two. You are a contestant on the show and get to choose a door; you will receive the prize behind your chosen door. Before revealing your prize, Monty opens one of the other doors to reveal a goat, and then he offers you the chance to switch your choice to the other unopened door. Should you switch? Many people are surprised to discover that, under reasonable assumptions that we will discuss below, you should switch. For a thorough discussion of the problem, see [**129**, Monty Hall problem].

Now suppose there are four doors, with a car behind one door and goats behind the other three. Once again you will win the prize behind the door of your choice. After you choose, Monty will open a door you didn't choose that hides a goat and offer you the chance to switch to a different door. Then, whether you switch or not, Monty will reveal another goat behind a door different from your choice at that time. Once again, you have a chance to switch before your prize is revealed. What should you do?

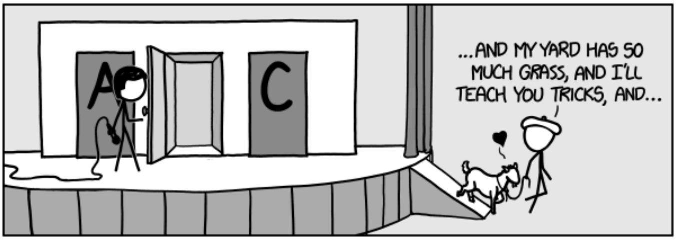
We make the following assumptions:

(1) Before the game begins, the prizes are distributed behind the doors at random.

(2) Monty knows which door hides the car, and he always chooses to open a door hiding a goat.

(3) When there is more than one door that Monty could open to reveal a goat, he chooses which to open at random.

And, of course, we also assume that you would rather win a car than a goat!



https://xkcd.com/1282/

7

Algorithms and Strategy

69 Where's Bob?

Agent Alice is on the trail of criminal hacker Bob and knows that he is hiding in one of 17 caves. The caves form a linear array and Bob moves only at night: every night he moves from the cave he is in to one of the nearest caves on either side of it. Alice can search two caves each day, with no restrictions on her choice. For example, if Alice searches $\{1, 2\}, \{2, 3\}, \ldots, \{16, 17\}$, then she will find Bob, but it might take as long as 16 days. Find a search strategy for Alice that is guaranteed to find Bob in as few days as possible.

70 It's a Horse Race

You are arranging races for 25 horses on a track that can handle only five horses at a time. How many races are needed to determine the three fastest horses, in order? The conditions are:

- Each horse always runs the distance in the same time.
- The 25 times are all different.
- You have no stopwatch, but can observe the finishing order in each race.

71 Your Two Best Shots

A perfectly camouflaged tank occupies one square in a field that is partitioned into a 15×15 grid of 225 squares. Your weapon can fire shots, each of which destroys one square. Two shots are necessary to destroy the tank. After the first hit, and only then, the tank, still invisible, moves to an adjacent square in the horizontal or vertical direction. What is the minimal number of shots needed to guarantee its destruction?

72 Determine the Martian Majority

You have just landed on Mars where you find seven identical-looking spheres. From prior missions, you know the following:

- Each sphere is colored either red or blue on the inside; the color is not visible from the outside.
- When two spheres touch each other, they glow if and only if they are the same color.

Your job is to carry out as few comparisons as possible, with the goal of identifying a sphere that you are certain has the majority color. What is the smallest number of comparisons that is guaranteed to work?

73 Majority Rules

A sequence of 1000 names (not all distinct) is going to be read out to you, and you are told that one name is in the majority, that is, it occurs at least 501 times. You have a counter that starts at 0 and that you can increment or decrement whenever you hear a name. But your memory is very limited: you can remember only a single name at any given time (but when you hear a name you can remember it long enough to check if it is the same as the one in your memory, and you can then decide whether or not to replace the name in your memory with the new name).

Find a strategy that, after all names are read, will allow you to declare the name of the majority person.

74 Going for Gold

Charlie has 64 identical-looking gold-colored coins; all are counterfeit except one, which is pure gold. He calls Alice into his office and seats her at a table containing an 8×8 checkerboard. He places a coin on each square, with each one showing either a head or tail as he chooses. Charlie tells Alice which coin is made of gold. Alice may then flip one of the coins over, replacing it on its square, or do nothing. She then leaves the room and Bob enters and is seated in the same position, facing the board. He may take one of the coins. Find a joint strategy that allows Bob to always select the gold coin. Alice and Bob know the rules and can plan a strategy, but cannot communicate after Alice enters the room.

75 The Mensa Correctional Institute

Alice and Bob, prisoners at the Mensa Correctional Institute, are under the care of warden Charlie. Alice will be brought into Charlie's office on Sunday and shown a standard deck of 52 cards, face up in a row in some arbitrary order. Alice may, if she wishes, interchange two cards. She then leaves and Charlie turns each card face down in its place. Bob is then brought in and Charlie calls out a random target card, C. Bob can turn over cards, one after another, at most 26 times as he searches for the target. If he eventually turns over C, both prisoners are freed; if he fails to find the target, they stay in prison. Alice and Bob know the rules and are allowed to plan a strategy in advance. On Sunday they cannot communicate; Alice has no idea what the target card will be.

(a) Find a strategy that guarantees success for the prisoners regardless of Charlie's choices.

(b) Same setup as in (a), except that the deck has only five cards and Charlie's initial arrangement and choice of target are both purely random; Alice can interchange any two cards, or do nothing. Alice and Bob are set free if and only if Bob finds C by looking at only one card. Find the strategy for the prisoners that has the highest probability of success.

76 Matching Hats

Alice and Bob are prisoners of warden Charlie, who will place a stack of infinitely many hats on each of their heads. Each stack of hats is numbered from bottom to top with positive integers, and each hat is colored black or white, chosen randomly with equal probability. Each prisoner will see the colors of all hats on the other prisoner, but not his or her own. On Charlie's signal, each prisoner will write down a positive integer. Charlie will check the color of each prisoner's hat in the location specified by his or her declared number. Alice and Bob are freed if and only if the two chosen hats are the same color.

The prisoners know the rules and can plan a strategy in advance, but cannot communicate after the hats are placed. If they each make a choice independent of what they see (e.g., they both declare 1), then their probability of success is 1/2. Find a strategy that succeeds more than two-thirds of the time.

77 One Hat Too Many

There are ten prisoners, Alice, Bob, Charlie, ..., Jill. The warden will call them together and line them up in order, with Alice first. He has 11 differently colored hats—the colors are known to the prisoners—and will randomly place one hat on each prisoner's head, discarding the unused hat. Each prisoner can see the hats of only the prisoners in front of them: Alice sees all (except her own), Bob sees eight hats, Jill sees nothing. At some point Alice will guess her color; all the other prisoners can hear her guess. Then Bob will guess his hat color, and so on to Jill. If all prisoners correctly identify the colors of their hats, they will be freed. The prisoners know the rules in advance and can devise a strategy; no communication among them is possible once the hats are placed. Find a strategy that maximizes their probability of success.

78 The Prisoners Must Agree

Alice and Bob are prisoners of warden Charlie, who has a supply of red hats and blue hats. On the appointed day, the warden will place a hat on each prisoner's head. The hats might or might not be the same color: Charlie can place the hats any way he likes. Each prisoner will see the other prisoner's hat, but not his or her own.

After the hat placement, on Charlie's signal, the prisoners must simultaneously shout a color. They will be freed if they shout out the same color, and at least one of them has a hat of that color. After the hats are placed but before Charlie's signal, the prisoners can access a random number generator, which will display to both of them a random integer in some range they choose. The prisoners know the rules and can plan a strategy before the hat placement; they cannot communicate after the hats are placed and before they shout.

The prisoners' probability of success will depend on how Charlie places the hats. For example, suppose the prisoners use a random integer from $\{1, 2\}$ and agree to shout "red" if the integer is 1 and "blue" if it is 2. If Charlie puts hats of different colors on their heads then their success is guaranteed, but if he gives them hats of the same color then their probability of success is only 50%. We will say that their *guaranteed success rate* is the minimum probability of success, over all possible hat placements. Thus, in the example just given, the guaranteed success rate is 50%.

(a) Find a strategy that gives the prisoners a guaranteed success rate of more than 50%. What is the largest guaranteed success rate they can achieve?

(b) Same problem for 100 prisoners, where the warden can choose from 100 hat colors for each prisoner. If the prisoners shout out the same random color, their chance of success when Charlie uses the same color for everyone is 1%. Find a strategy that gives the prisoners a guaranteed success rate of more than 50%. (As before, the prisoners must shout out the same color and the color must appear on at least one of their heads for them to be freed. The prisoners have joint access to a random number generator.)

79 How to Use Irrelevant Information

Prisoners Alice and Bob will be freed if they can pass a certain test. A room contains a countably infinite collection of boxes, each labeled with a positive integer; inside each box is a real number. On the day of the test, Alice will enter the room and open all boxes except one. She then tells the warden what she thinks is inside the closed box and leaves the room. Then the boxes are closed, and Bob enters the room and follows the identical protocol. The warden will free the prisoners if at most one of them guesses incorrectly.

(a) Suppose there is a requirement that all but finitely many boxes must contain the number 0. Find strategies that Alice and Bob can follow to guarantee their freedom.

(b) Suppose that there is no restriction on the numbers in the boxes. Show that, even in this case, there are strategies for Alice and Bob that guarantee their freedom.

The prisoners know the rules and can plan their strategies beforehand but cannot communicate on test day. We assume that the prisoners are capable of performing actions involving infinitely many steps. For example, each prisoner is capable of opening infinitely many boxes, fully reading the real numbers in those boxes, and specifying any real number with complete precision as his or her guess. Also, the prisoners can agree on an infinite amount of information in their strategy session. A prisoner need not specify at the beginning of his or her turn which box is to remain unopened; the prisoner may open some boxes and use the numbers in those boxes to decide what further boxes to open and which box to leave unopened.

80 Flipping Pennies

Consider the following penny-flipping game. We start with n pennies arranged in a straight line, where $n \geq 2$. A move consists of choosing a penny and turning it over (from heads to tails or vice versa) and doing the same to each of its neighbors. If the penny is at one end of the line, then it has only one neighbor; otherwise, it has two. For example, if the heads-tails sequence is HHTH and you choose the third penny, then your move would result in HTHT. For which values of n is it always possible to find a sequence of moves to bring all coins to heads, no matter what the initial state of the coins is?

81 Battleship Destruction

A battleship of length 2 is moving along the real line. At time 0 its center is at point X and it then moves with constant speed V, but both X and V are unknown integers; it is also not known whether it is moving left or right. At every second you can shoot at some point on the line. Find a strategy that guarantees that you will strike the ship in finite time. Assume that the projectile you shoot takes zero time to arrive at its target.

82 Real Battleship Destruction

A battleship is moving as in Problem 81, but now X and V are unknown real numbers. Find a strategy that guarantees that you will strike the ship in finite time.

83 A Very Local Maximum

You have 55 cards arranged in a circle. On the underside of each card is a real number, all different from each other and unknown to you. You wish to find a local maximum: a card whose number is larger than the numbers on the two adjacent cards.

What is the smallest number of cards that need to be turned over in order to be guaranteed of finding such a card? (The choice of which card to turn over may depend upon the results of the previous turns.)

84 Detecting a Black Hole

Imagine a network of 32 nodes and a communication system that allows messages to be sent from node i to node j for only some of the pairs (i, j). For any pair (i, j) you can, with one query, learn whether direct communication from i to j is possible. You want to use a sequence of such queries to find, if possible, a black hole: a node such that every other node can send messages directly to it, but it cannot send messages to any other node. What is the smallest number of queries you can make that is guaranteed to either find a black hole or determine that there is none?

85 Pablito's Solitaire

Pablito's Solitaire is played with checkers on an infinite triangular board of hexagons, as shown in Figure 85.1. The hexagon at the top forms row 1 of the board. Row 2 has two hexagons, row 3 has three hexagons, and so on. To play the game, you place any finite number of pieces at or below some row R, and you jump and remove pieces as in checkers (a piece jumps over an occupied neighboring hexagon to an open hexagon, and the jumped piece is removed). The goal is to end with a single piece in the top cell. An example with $R = 4$ is shown in Figure 85.2. What is the largest value of R for which the goal can be achieved?

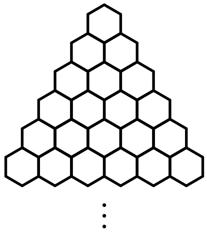


Figure 85.1. The board for Pablito's Solitaire. The board continues downward, with infinitely many rows.

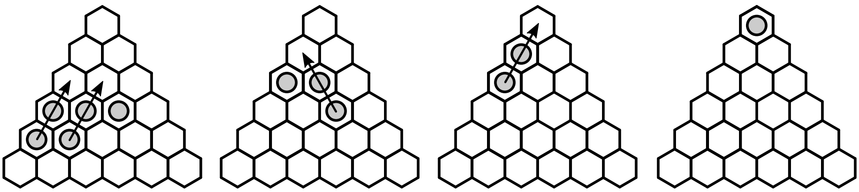


Figure 85.2. An example of playing Pablito's Solitaire, with $R = 4$.

This problem is due to Pablo Guerrero-García, Malaga, Spain.

86 Are the Coins Authentic?

You have ten coins. You know that at least one is authentic, but some of them may be counterfeit. Counterfeit coins have a different weight than authentic coins, but all authentic coins have the same weight and all counterfeit have the same weight. Show how to use an equal-arm balance three times so that you can determine whether there are any counterfeit coins. Note that an equal-arm balance can be used to compare two collections of coins to determine whether they weigh the same amount and, if not, which collection is heavier, but it does not tell you how much either collection of coins weighs.

87 Web Site Analysis

You have just launched a web site with a ton of traffic but very little money. You want to analyze your site's users, but you can afford to store information for only 1000 users. Therefore you decide to sample users uniformly over the lifetime of the site. You do not know how many users will ultimately appear.

All users up to the thousandth must be sampled, because there might not be more than 1000 users. But once that limit is surpassed you want to sample in such a way that each of the n users will have an equal probability of landing in the sample. You must make a one-time decision about whether or not to sample a user the moment he or she first visits the site. If you choose to include a new user in the sample, you must delete one user currently in the sample. Find a protocol that, assuming there are n users in all, guarantees that each of the n people has an equal probability of being in the final sample.

88 Find the Car, and the Car Key

Behind three doors labeled 1, 2, 3 are randomly placed a car, a key, and a goat. Alice and Bob win the car if Alice finds the car and Bob finds the key. First Alice (with Bob not present) opens a door; if the car is not behind it she can open a second door. If she fails to find the car, they lose. If she does find the car, then all doors are closed and Bob gets to open one of the three doors in the hope of finding the key and, if not, trying again with a second door. Alice and Bob do not communicate except to make a plan beforehand. What is their best strategy?

89 Magic Coins

You are arrested in a country with an unusual judicial system. After arriving at the prison you are given eight coins. You are told that four of them are magic, but to you they all look identical. Your jailers, however, can easily distinguish the magic coins from the others.

Once each day, you may choose a subset of the eight coins. If your set contains exactly two magic coins, you will be immediately released. What is the smallest R such that you can ensure you will be released in R days?

90 Russian Cards

Charlie has seven cards numbered 1, 2, 3, 4, 5, 6, 7 and randomly deals three to Alice and three to Bob (all face down), keeping the remaining card for himself. All three people can see the values of only the cards they hold. Show how Alice can make a public statement—one that she knows to be true—so that Bob can use it to learn her cards, but Charlie does not learn the location of any card except his own. Alice and Bob have not met prior to the deal, and so cannot have agreed on any sort of code. All three know the rules and the method of dealing the cards.

91 The Generous Automated Teller Machine

You have n boxes: B_1, B_2, …, B_n. Initially, B_1 contains one coin and the other boxes are empty. You may make moves of the following types:

- add_k: Remove a coin from B_k and add two coins to B_{k+1}. This move is allowed only if $1 \le k \le n-1$ and B_k is nonempty.
- flip_k: Remove a coin from B_k and switch the contents of B_{k+1} and B_{k+2}. This move is allowed only if $1 \le k \le n-2$ and B_k is nonempty.

For example, if $n = 4$, then applying the moves add_1, add_2, add_3, and flip_2 would give the following results (the lists of numbers give the number of coins in each box):

$$(1,0,0,0) \underset{\mathrm{add}_1}{\longrightarrow} (0,2,0,0) \underset{\mathrm{add}_2}{\longrightarrow} (0,1,2,0) \underset{\mathrm{add}_3}{\longrightarrow} (0,1,1,2) \underset{\mathrm{flip}_2}{\longrightarrow} (0,0,2,1).$$

(a) If $n = 5$, what is the largest number of coins you can get into the boxes by using these moves?

(b) Suppose $n = 6$. Show how to use these moves to get more than one million coins into the boxes. Can you get 10^{100} coins?

Photo of Dan Velleman courtesy of Shelley Velleman.

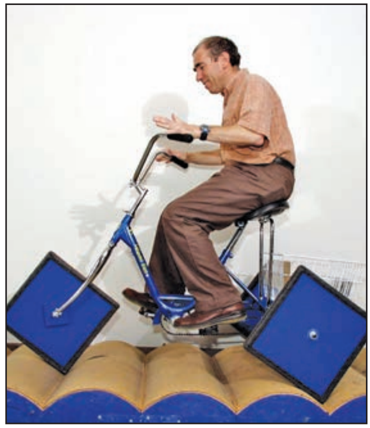


Photo of Stan Wagon courtesy of Macalester College.

Bicycle or Unicycle? is a collection of 105 mathematical puzzles whose defining characteristic is the surprise encountered in their solutions. Solvers will be surprised, even occasionally shocked, at those solutions. The problems unfold into levels of depth and generality very unusual in the types of problems seen in contests. In contrast to contest problems, these are problems meant to be savored; many solutions, all beautifully explained, lead to unanswered research questions. At the same time, the mathematics necessary to understand the problems and their solutions is all at the undergraduate level. The puzzles will, nonetheless, appeal to professionals as well as to students and, in fact, to anyone who finds delight in an unexpected discovery.

These problems were selected from the Macalester College Problem of the Week archive. The Macalester tradition of a weekly problem was started by Joseph Konhauser in 1968. In 1993 Stan Wagon assumed problem-generating duties. A previous book written by Wagon, Konhauser, and Dan Velleman, *Which Way Did the Bicycle Go?*, gathered problems from the first twenty-five years of the archive. The title problem in that collection was inspired by an error in logic made by Sherlock Holmes, who attempted to determine the direction of a bicycle from the tracks of its wheels. Here the title problem asks whether a bicycle track can always be distinguished from a unicycle track. You'll be surprised by the answer.

ISBN 978-1-4704-4759-5

9 781470 447595

PRB/36

For additional information and updates on this book, visit
www.ams.org/bookpages/prb-36

AMS
www.ams.org