January 11, 2019
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A very interesting problem. Sort of like the open Collatz problem in some ways. You know, no matter where you start the iteration, you end up at the same place.

Finding solutions that take 7 iterations to complete or 10 iterations to complete is not hard. For example, (80,25,32) for 7 iterations and (9074,1335,2318) for 10 iterations.

Making sense of this without random simulation to find solutions is more difficult. First, some results that are useful.

Lemma 1. After any iteration, the "raw" values of x, y, z—that is, before taking absolute values—satisfy $x–y = y–z$.

Proof. The system of equations is

(1) $x_{n+1} = 2x_n + y_n - 3z_n$
(2) $y_{n+1} = x_n + 3y_n - 4z_n$
(3) $z_{n+1} = 5y_n - 5z_n$

Subtracting (2) from (1) and (3) from (2) give $x_{n+1} - y_{n+1} = x_n - 2y_n + z_n = y_{n+1} - z_{n+1}$.
QED

Lemma 2. Adding a constant c to each of the starting positive integers x, y, and z does not change the result of an iteration.

Proof. The sum of the coefficients on the right-hand-sides of (1), (2), and (3) is zero.
QED

Lemma 3. If x,y,z after applying equations (1), (2), and (3), but before taking absolute values, are all positive or all negative, then at most two further iterations yields 0,0,0.

Proof. If all are negative, then all will be positive after applying absolute values. Thus, without loss of generality, we make take all to be positive and by Lemma 1, we can have $y_n = x_n + b$, $z_n = x_n + 2b$ for some integer b, positive, negative, or zero. Then from (1), (2), and (3) we obtain

(4) $x_{n+1} = 2x_n + y_n - 3z_n = -5b$
(5) $y_{n+1} = x_n + 3y_n - 4z_n = -5b$
(6) $z_{n+1} = 5y_n - 5z_n = -5b$

If $b = 0$, we have achieved 0,0,0. If $b \neq 0$, then one further iteration yields 0,0,0 because $x_{n+1} = y_{n+1} = z_{n+1}$.
QED

Armed with these results, we can make progress on the general problem. We start with a triplet of non-negative integers x_0,y_0,z_0. The case when all three are equal has been handled in the proof of Lemma 3. The cases when two of the three are equal can be viewed as limiting cases of the general case for x,y,z all distinct. In the general case, one of x_0, y_0, and z_0 is the smallest. Let the constant c of Lemma 2 be the negative of the smallest value. Then without loss of generality, we have three cases to consider:
(a) $x_0 = 0$, (b) $y_0 = 0$, and (c) $z_0 = 0$.

Case (a): $x_0 = 0$.

Equations (1), (2), and (3) become

(7) $x_{n+1} = y_n - 3z_n$
(8) $y_{n+1} = 3y_n - 4z_n$
(9) $z_{n+1} = 5y_n - 5z_n$

If $y_n > 3z_n$, the all of x_{n+1}, y_{n+1}, and z_{n+1} are positive and if $y_n < z_n$, then all of x_{n+1}, y_{n+1}, and z_{n+1} are negative. In either of those situations Lemma 3 applies. If $z_n < y_n < 3z_n$, then $x_{n+1} < 0$ and $z_{n+1} > 0$; y_{n+1} is negative if $y_n < 4z_n/3$ and positive if $y_n > 4z_n/3$.

Case (b): $y_0 = 0$.

Equations (1), (2), and (3) become

(10) $x_{n+1} = 2x_n - 3z_n$
(11) $y_{n+1} = x_n - 4z_n$
(12) $z_{n+1} = -5z_n$

From (12), we see that z_{n+1} is negative. Both x_{n+1} and y_{n+1} are negative if $x_n < 3z_n/2$. In that situation, Lemma 3 applies. If $x_n > 4z_n$, then both x_{n+1} and y_{n+1} are positive, while if $3z_n/2 < x_n < 4z_n$, x_{n+1} is positive and y_{n+1} is negative.

Case (c): $z_0 = 0$.

Equations (1), (2), and (3) become

(13) $x_{n+1} = 2x_n + y_n$
(14) $y_{n+1} = x_n + 3y_n$
(15) $z_{n+1} = 5y_n$

All of x_{n+1}, y_{n+1}, and z_{n+1} are positive and Lemma 3 applies.

These three cases identify situations in which the iterations will terminate after either one or two additional steps. They also identify situations in which the iterations will continue.

After each iteration where the process continues, the smallest of x_{n+1}, y_{n+1}, and z_{n+1} is determined and Lemma 2 applied with the constant c chosen as the negative of the smallest value, leading once again to new Cases (a), (b), and (c).

From the above, we see that if z is the smallest among the starting x, y, and z, then the process ends after at most three iterations. For it to continue beyond three iterations, either x or y must be the smallest. It is instructive to examine two examples that complete after iteration 7. In the first, x is the smallest, and in the second, y is the smallest.

<u>First example:</u>

```
ITERATION: 1
STARTING VECTOR:               573     870     832
MODIFIED STARTING VECTOR:        0     297     259        CASE (A)
ENDING VECTOR:                ¯480    ¯145     190        CONTINUE
ABSOLUTE VALUE:                480     145     190

ITERATION: 2
STARTING VECTOR:               480     145     190
MODIFIED STARTING VECTOR:      335       0      45        CASE (B)
ENDING VECTOR:                 535     155    ¯225        CONTINUE
ABSOLUTE VALUE:                535     155     225

ITERATION: 3
STARTING VECTOR:               535     155     225
MODIFIED STARTING VECTOR:      380       0      70        CASE (B)
ENDING VECTOR:                 550     100    ¯350        CONTINUE
ABSOLUTE VALUE:                550     100     350

ITERATION: 4
STARTING VECTOR:               550     100     350
MODIFIED STARTING VECTOR:      450       0     250        CASE (B)
ENDING VECTOR:                 150    ¯550   ¯1250        CONTINUE
ABSOLUTE VALUE:                150     550    1250

ITERATION: 5
STARTING VECTOR:               150     550    1250
MODIFIED STARTING VECTOR:        0     400    1100        CASE (A)
ENDING VECTOR:               ¯2900   ¯3200   ¯3500        LEMMA 3
ABSOLUTE VALUE:               2900    3200    3500

ITERATION: 6
STARTING VECTOR:              2900    3200    3500
ENDING VECTOR:               ¯1500   ¯1500   ¯1500
ABSOLUTE VALUE:               1500    1500    1500

ITERATION: 7
STARTING VECTOR:              1500    1500    1500
ENDING VECTOR:                   0       0       0
```

<u>Second example:</u>

```
ITERATION: 1
STARTING VECTOR:                80      25      32
MODIFIED STARTING VECTOR:       55       0       7        CASE (B)
ENDING VECTOR:                  89      27     ¯35        CONTINUE
ABSOLUTE VALUE:                 89      27      35

ITERATION: 2
STARTING VECTOR:                89      27      35
MODIFIED STARTING VECTOR:       62       0       8        CASE (B)
ENDING VECTOR:                 100      30     ¯40        CONTINUE
```

```
ABSOLUTE VALUE:                100      30      40

ITERATION: 3
STARTING VECTOR:               100      30      40
MODIFIED STARTING VECTOR:       70       0      10        CASE (B)
ENDING VECTOR:                 110      30     ¯50        CONTINUE
ABSOLUTE VALUE:                110      30      50

ITERATION: 4
STARTING VECTOR:               110      30      50
MODIFIED STARTING VECTOR:       80       0      20        CASE (B)
ENDING VECTOR:                 100       0    ¯100        CONTINUE
ABSOLUTE VALUE:                100       0     100

ITERATION: 5
STARTING VECTOR:               100       0     100
MODIFIED STARTING VECTOR:      100       0     100        CASE (B)
ENDING VECTOR:                ¯100    ¯300    ¯500        LEMMA 3
ABSOLUTE VALUE:                100     300     500

ITERATION: 6
STARTING VECTOR:               100     300     500
ENDING VECTOR:               ¯1000   ¯1000   ¯1000
ABSOLUTE VALUE:               1000    1000    1000

ITERATION: 7
STARTING VECTOR:              1000    1000    1000
ENDING VECTOR:                   0       0       0
```

The frequency of occurrences of the number of iterations required to reach 0,0,0 for a random experiment of 1,000,000 initial x,y,z, each positive integer drawn uniformly on [1,10000] is given below. No attempt was made to eliminate repetitions.

```
 1      0
 2     48
 3 778265
 4 192176
 5  25788
 6   3220
 7    434
 8     56
 9     12
10      1
```

The same experiment, but with the starting x,y,z drawn from [1,1000000], thus reducing the number of repetitions that might occur.

```
 1      0
 2      1
 3 777190
 4 193031
 5  25913
 6   3348
 7    462
 8     46
 9      8
10      1
```

The same experiment, but with the starting x,y,z drawn from [1,1000000000] to virtually eliminate the possibility of a repetition.

```
 1      0
 2      0
 3 777615
```

4	193019
5	25598
6	3295
7	404
8	58
9	10
10	1

If we rerun the last experiment, but with 10,000,000 trials instead of 1,000,000, we see the solitary occurrence of 11 iterations to complete.

1	0
2	0
3	7777253
4	1926034
5	258824
6	33054
7	4234
8	511
9	78
10	13
11	1

What appears to be going on in each case is that the number of iterations to complete depends on where in the relevant interval the starting values of x,y,z fall.

Case (a): With x = 0, where on the interval (z,3z) does y lie?
Case (b): With y = 0, where on the interval (3z/2,infinity) does x lie?

Lemma 4. With x, y, and z all positive integers, each uniformly distributed on [1,N], the probability that it takes exactly three iterations to reach 0,0,0 is 7/9 as N approaches infinity.

Proof. The following statements apply as N approaches infinity. The probabilities of cases (a), (b), and (c) are each 1/3. Given case (a), the probability that y < z is 1/2 and the probability that y > 3z is 1/6. Given case (b), the probability that x < 3z/2 is 2/3. Thus, the probability of exactly three iterations to reach 0,0,0 is 1/3 + (1/3)x[(1/2)+(1/6)] + (1/3)x(2/3) = 7/9.
QED

The last random experiment has sufficiently small error to indicate confirmation of Lemma 4. The same experiment is consistent with the following conjecture.

Conjecture 1. With $p_1 = p_2 = 0$ and $p_3 = 7/9$, the p_n for n > 3 is given by
$p_n = (13/15) \times [1 - \text{Sum from k=1 to k=n}-1 \text{ of } p_k]$

The formula for p_n for $n \geq 4$ can be simplified to $p_n = (13/15)\,(2/9)\,(2/15)^{(n-4)} = (13/9)\,(2/15)^{(n-3)}$ but is even more illuminating in the form originally written because it says that the probability that the process stops in exactly n iterations is equal to (13/15) of the probability that it has not stopped before n iterations. The reason that $p_1 = 0$ is that it requires all of the starting x,y,z to be equal and that is a probability 0 event as N approaches infinity. The reason that $p_2 = 0$ is that it requires the starting x,y,z to form an arithmetic sequence (constant difference) and that is a probability 0 event as N approaches infinity.

Stan, I look forward to seeing your full solution to this problem. I suspect there are good answers to how to divide up the interval (z,3z) for Case (a) and (3z/2,infinity) for Case (b) (in a way that involves only rational numbers) so that one knows right from the outset exactly how many iterations it will take to complete. Moreover, I suspect that there is a proved result to substitute for Conjecture 1 if that conjecture turns out to be false.

******************** END OF FIRST PART OF NOTE ********************

Now we turn to creating arbitrarily long sequences before the iterations halt at 0,0,0, quite likely in a manner different from the "formulas" you've already received. The analysis is based on the previous methodology: in particular, on Case (b).

We use the following notation. We use a superscript ** to indicate the "raw" output from applying the matrix to the starting column vector. We use a superscript * to indicate the absolute value of the raw output and we use no subscript to indicate a vector x,y,z that has been adjusted by adding a constant vector so that y = 0. We identify a starting vector that gives a PERPETUAL Case (b). Hence, in theory, it NEVER halts. But because x, y, and z have to be positive integers, it does halt in practice.

The Case (b) equations for (1), (2), and (3) are

(16) $x^{**}_{n+1} = 2x_n - 3z_n$
(17) $y^{**}_{n+1} = x_n - 4z_n$
(18) $z^{**}_{n+1} = -5z_n$

We start with the vector $x_n=az_n$, $y_n=0$, z_n where $a > 4$. Then $x^{**}_{n+1} > 0$, $y^{**}_{n+1} > 0$, and $z^{**}_{n+1} < 0$. Thus,

(19) $x^{*}_{n+1} = 2x_n - 3z_n$
(20) $y^{*}_{n+1} = x_n - 4z_n$
(21) $z^{*}_{n+1} = 5z_n$

Adding the constant $4z_n - x_n$ to each of equations (19), (20), and (21), we obtain

(22) $x_{n+1} = x_n + z_n$
(23) $y_{n+1} = 0$
(24) $z_{n+1} = -x_n + 9z_n$

To make the series perpetual, we need $x_{n+1} / z_{n+1} = a$. This leads to the equation

(25) $(a + 1) / (9 - a) = a$

This is a quadratic with two solutions. But only one satisfies a > 4. It is

(26) $a = (8 + \sqrt{60}) / 2 = 4 + \sqrt{15} = 7.87298334\ldots$

Since a is irrational and since x, y, and z must be positive integers, we cannot in practice obtain a perpetual sequence. However, we can obtain an arbitrarily long one by truncating sufficiently far out the decimal expansion for the number a. Then we let z be the smallest power of ten that results in (truncated a) times z being an integer.

If we truncate a after n decimals we get the following table for n = 0, 1, 2, … , 8:

7,0,1:	5 iterations
78,0,10:	6 iterations
787,0,100	8 iterations
7872,0,1000	9 iterations
78729,0,10000	10 iterations
787298,0,100000	11 iterations
7872983,0,1000000	12 iterations
78729833,0,10000000	13 iterations
787298334,0,100000000	14 iterations

I end with another conjecture.

Conjecture 2. The sequence for x,y,z above produces an associated sequence in the number of iterations until the process stops that is *non-decreasing* and *unbounded*.

This one ought to be pretty easy to prove, but I haven't tried.