Solution. Solutions were received from Philippe Fondanaiche, Dan Dima, Dan Velleman, Etan Basser, Jim Tilley, and Piotr Zielinski. The correct answer was received from Bob Bixler, Michael Elgersma, Ralph Dratman, and J. J. Cote.

There are several approaches to this problem. There are some relatively simple intuitive approaches that rely on unproved assumptions (such as: $y \to 0$ as $t \to \infty$). They work to get the right answer (which is 1/2), but they are not fully rigorous. There is also a very elegant approach using parabolas (Zielinski) that I will give later (but it requires knowledge of parabola properties). The source of the problem can be viewed at ⟨http://www.ma.huji.ac.il/hart/puzzle/pursuit.pdf?⟩

A blog that took a poll to see what intuition might lead to is at:

⟨https://gilkalai.wordpress.com/2018/06/29/test-your-intuition-35-what-is-the-limiting-distance⟩

In the poll, the choices 0 and 1 both beat out the correct answer, 1/2.

The classic approach at ⟨https://en.wikipedia.org/wiki/Radiodrome⟩ is rigorous, and gives more than just the answer, but requires using slightly more powerful tools (eg, second derivatives). So I will present here a solution by Dan Velleman that appears to be the simplest rigorous route to 1/2.

Let $B = (x, y)$; $A = (t, 0)$; $z = t - x$ and $s = \|A - B\| = \sqrt{z^2 + y^2}$. Here x, y, z, and s are functions of t; we use x' and similar to refer to differentiation with respect to t.

0. $s^2 = y^2 + z^2$

1. $(x', y') = \left(\frac{z}{s}, \frac{-y}{s}\right)$ (because B's speed is 1 and is toward A)

2. $s' = x' - 1 = -z'$. Proof: By (1), $2\,s\,s' = 2\,z\,z' + 2\,y\,y' = 2\,z\left(1 - \frac{z}{s}\right) + 2y\left(\frac{-y}{s}\right) = 2\,z - 2\,\frac{(y^2+z^2)}{s} = 2z - 2s$ and $s' = \frac{z}{s} - 1$. Using $z' = (t - x)' = 1 - x' = 1 - \frac{z}{s}$ from (1) then gives (2). This is the key fact. At first it seems like a miraculous coincidence. But in fact it can also be derived from simple vector geometry as in the figure and caption that follow.

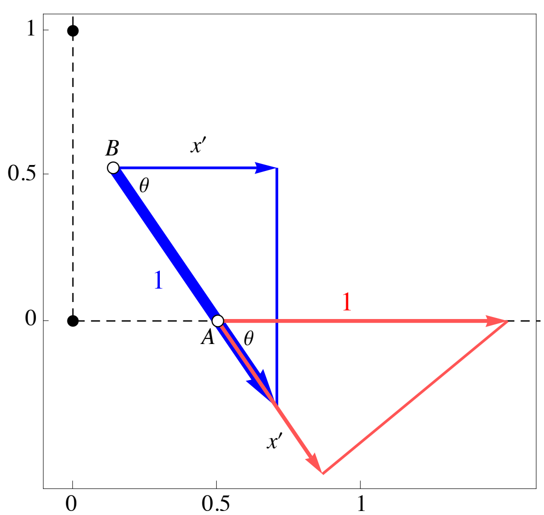


Figure 1. The horizontal blue arrow is the x-component of B's velocity. The red and blue triangles are congruent. Therefore the angled red arrow, which is the component of A's velocity in the direction away from B, has length equal to that of x'. It follows that $s' = x' - 1$ because the change in s is the difference of the speed at which A moves away from B and the speed at which B moves toward A.

3. $s = 1 - z$, $z = 1 - s$ (integrate (2) and use the initial condition)

It is tempting now to say that because $\lim_{t\to\infty} y = 0$, (0) gives $s/z \to 1$ and (3) then gives $s \to 1/2$; this is the correct answer. But we do not yet know the limiting value of y. We next prove that $y \to 0$. This can be done directly, but the approach that follows based on s is more useful. We know that $s \geq 0$, and the DE for s below will prove useful later.

4. $s' = \frac{z}{s} - 1$ (by (2)).

5. s' is nonpositive, so s is nonincreasing (by (4) because $z \leq s$ (by (0)).

6. $0 \geq s' = \frac{z}{s} - 1 = \frac{1-s}{s} - 1 = \frac{1}{s} - 2$ (by (4), (5)) so $s \geq 1/2$.

7. The limiting value s_∞ exists and $\frac{1}{2} \leq s_\infty \leq 1$. (By (5), (6), and the fact that a nonincreasing function that is bounded from below has a limit.)

8. $\lim_{t\to\infty} s' = \frac{1}{s_\infty} - 2$ (by (6)) so this limit exists. But this limit must be 0, since any other value implies that $s \to \pm\infty$. And $0 = \frac{1}{s_\infty} - 2$ means $s_\infty = \frac{1}{2}$.

This solves the problem, but we can now derive several relationships and so get a formula for the pursuit curve.

9. $t = \frac{1}{4}(2 - 2s - \ln(2s - 1))$

10. $t = \frac{1}{4}\left(1 - y^2 - 2\ln y\right)$ (by (9) and (3))

11. $x = \frac{1}{4}\left(y^2 - 1 - 2\ln y\right)$. This gives a simple formula that defines the pursuit curve as a graph. But the next equations give x and y as functions of t.

12. $y = \sqrt{W\left(e^{1-4t}\right)}$ and $x = \frac{1}{2}\left(W\left(e^{1-4t}\right) + 2t - 1\right)$, where W is Lambert's W-function (see [http://mathworld.wolfram.com/LambertW-Function.html]).

So 12 gives a representation of the motion in terms of a venerable function from a century and a half ago. To prove these:

Proof of 9. The differential equation $s' = \frac{1}{s} - 2$ becomes $\frac{s}{1-2s}\,ds = dt$, which is easily integrated; the condition $s(0) = 1$ then gives (9). Note that we know $s \geq 1/2$ but (9) implies $s > \frac{1}{2}$. I don't think we need to know that $s > 1/2$ to get (9). The preceding integration tells us that $t(s) = \frac{1}{4}(2 - 2s - \ln(2s - 1))$ for $1/2 < s \leq 1$. And we know from (5) that t cannot decrease as s decreases from 1. The formula means that $t \to \infty$ as $s \to 1/2$, so $s(t)$ cannot be $1/2$ at a finite value of t.

12. Inverting (9) gives $s = \frac{1}{2}\left(1 + W\left(e^{1-4t}\right)\right)$, Then $x = t - 1 + s = t - \frac{1}{2} + \frac{1}{2}W\left(e^{1-4t}\right)$. And, by (0), $y = \sqrt{(1 - t + x)^2 - (t - x)^2} = \sqrt{1 - 2t + 2x}$ (by (3)) so $y = \sqrt{W\left(e^{1-4t}\right)}$.

10. Substituting $s = \left(y^2 + 1\right)/2$ into (9) gives (10).

11. Add $\frac{y^2}{2} - \frac{1}{2}$ to both sides of (11) and use $y^2 = 1 - 2z$ to get (11).

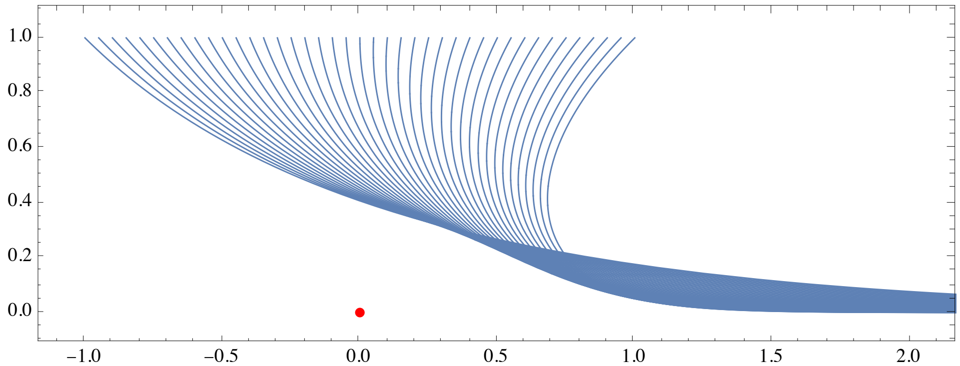
These methods lead to further results.

1. (found by Michael Elgersma, proved by Dan Velleman) If B starts at (x, y), then the limiting distance from A is $\left(\sqrt{x^2 + y^2} - x\right)/2$. If $y = 0$ and $x < 0$, then this is $-x$; if $x \geq 0$, this is 0, as B crashes into A.)

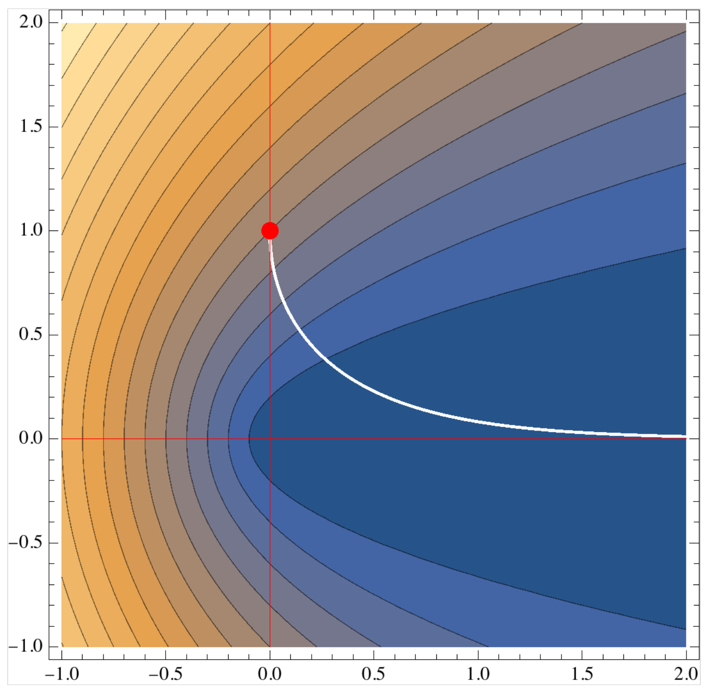
2. Velleman: Suppose B starts at (x_0, y_0). Then the analog of 11 is:

11A: $x = \frac{1}{4}\left(\frac{y^2}{C} - 2\,C\ln(y) + D\right)$, where $s_0 = \sqrt{x_0^2 + y_0^2}$, $C = s_o - x_0$, $K = \frac{-s_0}{2} - \frac{C}{4}\ln\left(s_0 - \frac{C}{2}\right)$, and $D = \frac{C}{4}\ln(2\,C) - 3\,\frac{C}{4} - K$.

Here is a plot of several pursuit curves starting from points with $y = 1$.

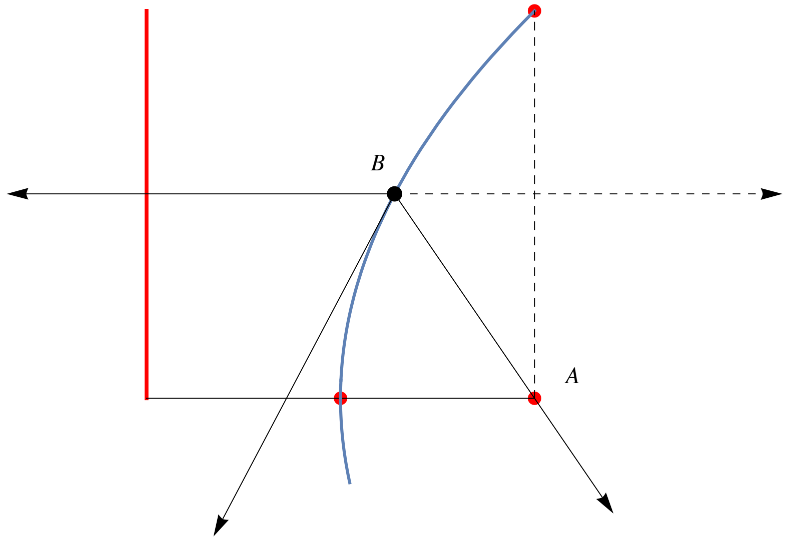


Here is a contour plot of the limiting distance for any start point in the plane, together with just the one pursuit curve.



This sort of pursuit problem is very old. The curve is known as a *dog curve* or *radiodrome*. It appears that this exact problem was first investigated in 1732. See ⟨https://en.wikipedia.org/wiki/Radiodrome⟩. A relatively recent book about pursuit curves is Paul Nahin, *Chases and Escapes*. *The Mathematics of Pursuits and Evasion*, Princeton University Press, 2012.

Finally, here is a nice geometric approach to the problem by Piotr Zielinski. Work in the frame of reference of A. Then B's velocity vector is this coordinate system is the sum of two unit vectors: $(-1, 0)$ because in the original coordinate system A is moving to the right, and a unit vector from B to A. This is exactly the condition of B's trajectory being a parabola with A at its focus. Because the distance between the directrix and the focus of the parabola equals the length of the "semilatus rectum" (which is the original distance from B to A), the limiting distance is $1/2$. There are several details to check to make this proof rigorous (e.g. that B does not stop before hitting the x-axis), but it all works. The diagram below shows how B's velocity vector bisects the two parts of its velocity, and this fact about angles is the defining reflective property of a parabola.



As always, thanks to all the solvers for their input.