Consecutive Integer Squares

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From Stan Wagon's Problem of the Week 1229

Problem Statement:

For which positive integers n is there a sum of n consecutive integers that is a perfect square?

Solution:

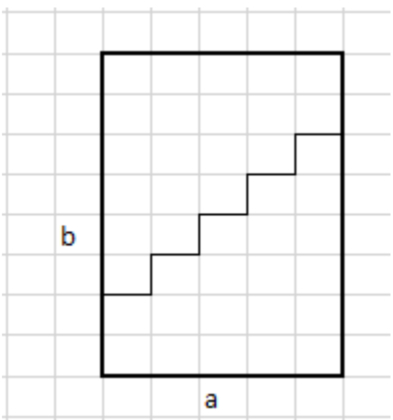
A natural approach is to calculate such a sum as an explicit formula. However we prefer to reframe the problem as one about rectangles and integer triples which avoids the need for much algebra. We will use one note of geometry and some simple logic as our main tools.

The complete general solution is given below as Theorem 1. The proof will come simply from our approach towards the end.

Theorem 1: Any given positive integer n factorises uniquely as $2^f g^2 h$ where g, h are odd and h is square free. The general solution for sequences of consecutive integers of length n that sum to a perfect square is given by:

$hr^2 - (n-1)/2 \;..\; hr^2 + (n-1)/2$	for any integer r,	when $f = 0$
$(hr^2 - (n-1))/2 \;..\; (hr^2 + (n-1))/2$	for any odd integer r,	when f odd
Non-existent		when f>0 even

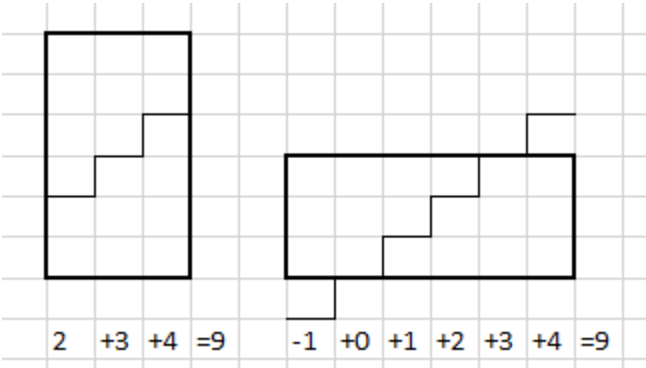
Consider a rectangle with width integer a, and height integer b>a. The rectangle is such that it can be split into identical halves using a stepped line composed of edges of the unit squares.



The centre of the rectangle is a centre of rotation for the stepped line, and so is in the middle of an edge of a unit square. If this is a horizontal edge (as shown) then *a* is odd and *b* even. If it is vertical then *a* is even and *b* odd. So certainly *a* and *b* have different parity.

Each half represents a sequence of consecutive numbers. If the area is $2c^2$ then it is our solution to the original problem for $n = a$.

Less obvious is that it is also a solution for $n = b$, if we allow the sequence to contain zero and negative integers. We illustrate this with an example:



In the second case, on the right of the diagram, the area above the rectangle is equal to the area below. Since the former contributes a positive amount to the sum and the latter contributes a negative amount they cancel out. So the sum of the sequence is half of the area of the rectangle.

Any sequence of consecutive integers will generate such a rectangle.

With this observation we may reframe the problem as follows.

We say that a triple of positive integers, (a, b, c), is a **solution** if it satisfies the following conditions:

1. *a* is odd
2. *b* is even
3. $ab = 2c^2$

Note that we have removed the condition that $b > a$.

We use the term solution since it represents solutions to the original problem for $n = a$ and $n = b$.

The example above is the solution (3, 6, 3) and so represents a solution to the original problem for n=3 and n=6.

Our next results are put in the form of a Lemma.

Lemma 1:

i. $(a, 2a, a)$ is a solution for any odd positive integer *a*.
ii. If (a, b, c) is a solution then
    a. (a, r^2b, rc) is a solution for any integer *r*,
    b. (r^2a, b, rc) is a solution for any odd integer *r*.
iii. In any solution *b* is divisible by 2 an odd number of times.
iv. There is a solution for *n* precisely when either *n* is odd or *n* is divisible by 2 an odd number of times.
v. If there is a solution for *n* then there is a solution where the consecutive integers are positive.

Proof of Lemma 1:

i. and ii. are simple to check against the conditions for a solution.

iii. follows from condition 3, that $ab = 2c^2$. We consider that the RHS is divisible by 2 an odd number of times and that a is odd.

For iv: If n is odd we use i. If n is even then we use i. and ii.a. with r a power of 2. The situation where n is divisible by 2 an even number of times is excluded by iii.

For v. we use ii. to get a rectangle with n the shortest side.

Lemma 1.iv solves the stated problem. Lemma 1.v shows that we only need consider positive integers.

The above demonstrates when a solution exists for n. We wish to calculate the sequences generated by (a, b, c). There are two such sequences, for $n = a$ and $n = b$.

Let s be the first number in a sequence and t be the last number in the sequence so that the sequence is $s..t$.

First take that case where $n=a$. We have the following equations to solve.

$t - s = a - 1$ since the length of the sequence is $n = a$

$t + s = b$ by the construction of the rectangle

from which

$t = (b + a - 1)/2$

$s = (b - a + 1)/2$

In the other case where $n=b$ then we must switch the values a and b to get

$t = (a + b - 1)/2$

$s = (a - b + 1)/2$

Note that t is the same in both cases. The difference in s is explained by observing that we add equivalent negative and positive integers, plus zero, to the start of the sequence.

Applying this to our example (3, 6, 3) we get $t = (6 + 3 -1)/2 = 4$. For $n=3$ we get $s= (6-3+1)/2 = 2$. For $n=6$ we get $s= (3-6+1)/2 = -1$. So the sequences are:

$2 + 3 + 4 = 9$

$-1 + 0 + 1 + 2 + 3 + 4 = 9$

A Worked Example

To show that this is a constructive solution let us consider a worked example.

Question: Can we find a sequence of 40 consecutive positive integers such that the sum is a perfect square?

Answer:

40 is $2^3 5$, so we start by using Lemma 1.i to get (5, 10, 5).

Next we use Lemma 1.ii.a to get (5, 40, 10).

As we want a positive solution we must have 40 the smallest of *a* and *b*, so we use Lemma 1.ii.b to get (45, 40, 30).

Using our formulae for *s* and *t* we find the sequence is (45-40+1)/2 .. (45+40-1)/2 which is 3..42.

We should check that the sum is 30^2.

$$\sum_{i=3}^{42} i = \sum_{i=1}^{42} i - \sum_{i=1}^{2} i$$

$$= \frac{42(42+1)}{2} - \frac{2(2+1)}{2}$$

$$= 903 - 3$$

$$= 30^2$$

The General Solution

Any positive integer *n* with a solution can be factorised uniquely as $n = 2^f g^2 h$ where *f* is zero or odd, *g* and *h* are odd and *h* is square-free.

First we consider the case where *n* is odd, which occurs when $f = 0$ and so $n = g^2 h$.

First we find the general solution in terms of triples. A triple representing a solution must have $a = n = g^2 h$, since n is odd. Then *b* must be divisible by $2h$ to satisfy $ab = 2c^2$. That gives the general solution $(g^2 h, 2r^2 h, rgh)$ for any integer *r*.

The sequence is given by

$s = (b - a + 1)/2 = (2r^2 h - g^2 h + 1)/2 = hr^2 - (n-1)/2$

$t = (b + a - 1)/2 = (2r^2 h + g^2 h - 1)/2 = hr^2 + (n-1)/2$

So the sequence is

$hr^2 - (n-1)/2$ .. $hr^2 + (n-1)/2$

In the other case *n* is even, which occurs when $f > 0$. From the Lemma 1 we know that *f* must be odd.

Again we find the general solution in terms of triples. A triple representing a solution must have $b = n = 2^f g^2 h$, since n is even. Then *a* must be divisible by *h* to satisfy $ab = 2c^2$. We have the general solution $(r^2 h, 2^f g^2 h, 2^{(f-1)/2} rgh)$ for any odd integer *r*.

The sequence is given by

$s = (a - b + 1)/2 = (r^2h - 2^f g^2 h + 1)/2 = (hr^2 - (n-1))/2$
$t = (a + b - 1)/2 = (r^2h + 2^f g^2 h - 1)/2 = (hr^2 + (n-1))/2$

So the sequence is

$(hr^2 - (n-1))/2 .. (hr^2 + (n-1))/2$

This proves Theorem 1 which we restate here for convenience.

Theorem 1: Any given positive integer n factorises uniquely as $2^f g^2 h$ where g, h are odd and h is square free. The general solution for sequences of consecutive integers of length n that sum to a perfect square is given by:

$hr^2 - (n-1)/2 .. hr^2 + (n-1)/2$	for any integer r,	when $f = 0$
$(hr^2 - (n-1))/2 .. (hr^2 + (n-1))/2$	for any odd integer r,	when f odd
Non-existent		when f>0 even

A Table of Solutions for small *n*

Using the formulae in Theorem 1 we can write a table of possible solutions for each n.

n	*f*	*g*	*h*	*From*	*To*	*Sqrt Sum*	*r* is …
1	0	1	1	r^2	r^2	r	
2	1	1	1	$(r^2-1)/2$	$(r^2+1)/2$	r	Odd
3	0	1	3	$3r^2-1$	$3r^2+1$	$3r$	
4	2	1	1				None Exist
5	0	1	5	$5r^2-2$	$5r^2+2$	$5r$	
6	1	1	3	$(3r^2-5)/2$	$(3r^2+5)/2$	$3r$	Odd
7	0	1	7	$7r^2-3$	$7r^2+3$	$7r$	
8	3	1	1	$(r^2-7)/2$	$(r^2+7)/2$	$2r$	Odd
9	0	3	1	r^2-4	r^2+4	$3r$	
10	1	1	5	$(5r^2-7)/2$	$(5r^2+7)/2$	$5r$	Odd
11	0	1	11	$11r^2-5$	$11r^2+5$	$11r$	
12	2	1	3				None Exist
13	0	1	13	$13r^2-6$	$13r^2+6$	$13r$	
14	1	1	7	$(7r^2-13)/2$	$(7r^2+13)/2$	$7r$	Odd
15	0	1	15	$15r^2-7$	$15r^2+7$	$15r$	
16	4	1	1				None Exist
17	0	1	17	$17r^2-8$	$17r^2+8$	$17r$	
18	1	3	1	$(r^2-17)/2$	$(r^2+17)/2$	$3r$	Odd

19	0	1	19	$19r^2$-9	$19r^2$+9	19*r*	
20	2	1	5				None Exist
21	0	1	21	$21r^2$-10	$21r^2$+10	21*r*	
22	1	1	11	($11r^2$-21)/2	($11r^2$+21)/2	11*r*	Odd
23	0	1	23	$23r^2$-11	$23r^2$+11	23*r*	
24	3	1	3	($3r^2$-23)/2	($3r^2$+23)/2	6*r*	Odd
25	0	5	1	r^2-12	r^2+12	5*r*	
26	1	1	13	($13r^2$-25)/2	($13r^2$+25)/2	13*r*	Odd
27	0	3	3	$3r^2$-13	$3r^2$+13	9*r*	
28	2	1	7				None Exist
29	0	1	29	$29r^2$-14	$29r^2$+14	29*r*	
30	1	1	15	($15r^2$-29)/2	($15r^2$+29)/2	15*r*	Odd
31	0	1	31	$31r^2$-15	$31r^2$+15	31*r*	
32	5	1	1	(r^2-31)/2	(r^2+31)/2	4*r*	Odd
33	0	1	33	$33r^2$-16	$33r^2$+16	33*r*	
34	1	1	17	($17r^2$-33)/2	($17r^2$+33)/2	17*r*	Odd