Solution to 1223 The Evil Warden.

This is one of those very rare PoWs (I cannot think of another case) that no one solved. About 10 of you submitted the basic approach, which gives a probability of 47%. I was shocked when I found that there is a method that works with probability 48%. All of this (and extensions of an early PoW in the same vein) leads to many research questions, and I have been working very hard on this with Carter and Rickert over the past few weeks. More info on request.

The problem: Problem 1223 The Evil Warden Alice and Bob are prisoners of warden Charlie. Alice will be brought into Charlie's room on Sunday and shown 5 cards, numbered 1, 2, 3, 4, 5, face-up in a row in a random order. Alice can, if she wishes, interchange two cards. She then leaves the room and Charlie turns all cards face-down in their places. Bob is then brought to the room. Charlie calls out a random target card T. Bob is allowed to turn over ONE CARD ONLY and if, and only if, he finds the target T, the two prisoners are freed. The odds of success seem poor. What is the prisoners' best strategy? Express the probability of success as n% where n is the nearest integer to the actual probability of your strategy. Note that Charlie's two choices -- the initial shuffle and the choice of target -- are assumed to be purely random. Source: Larry Carter, Mark Rickert, and Stan Wagon, who have been working on many variations of this problem, finding several surprises.

Solution. It is helpful to think of the cards as being behind doors 1, 2, 3, 4, or 5. The standard approach to such problems is to use a cycle-splitting strategy: Bob, on hearing T, opens door T. Let us call this strategy (12345), which indicates the door Bob opens on hearing the target T. Alice, on seeing the shuffle, looks to see if there are any transpositions. If so, she chooses one and switches the two cards. If not, she looks for a cycle of length K ($K \geq 3$) and splits it into a fixed point and a cycle of length $K - 1$. If there are no cycles, then the cards are in perfect order and she does nothing. Another way of saying this is: **Bob opens door *T*.** Alice does, as she always does for any strategy, the best thing she can for Bob's overall performance on the permutation he will see.

This basic cycle-splitting strategy has 284 good cases over all Charlie's choices, for a success probability of $\frac{284}{5 \cdot 5!} = 47\frac{1}{3}\%$. Here is the general formula for n cards.

Proof. Let $T(n)$ be the number of permutations of $\{1, \ldots, n\}$ having at least one transposition. A formula for this is on OEIS: https://oeis.org/A027616, which cites A000266. It is $T(n) = n! \sum_{k=1}^{n/2} (-1)^{k+1} \frac{1}{2^k\, k!}$, and it is easy derived by standard inclusion-exclusion arguments. For our case of n = 5, one can easily count: the three types for the permutations with a transposition are (23), (221), (2111) with frequencies 20, 10, 15, resp., for a total of $T(5) = 45$. In general, starting from $n = 1$, T-values are 0, 1, 3, 9, 45, 285, 1995. Let $f(n)$ be the number of successes using the basic cycle-splitting strategy.

Theorem. $f(n) = 2 \cdot n! - 1 + T(n)$

Proof. Among all $n!$ possibilities for Charlie's shuffle, the number of fixed points is $n!$ (each i is a fixed point of $(n-1)!$ permutations, so $n\,(n-1)!$ in all). Also, for any Charlie shuffle (except the identity), Alice's switch can always introduce at least one more fixed point, so that is another $n! - 1$ cases. But when there is a transposition, Alice can undo it, so that her move adds two to the count instead of just one. QED

So $f(5) = 240 - 1 + 45 = 284$ and success rate is $\frac{284}{5 \cdot 5!} = \frac{71}{150} = 47\frac{1}{3}\%$, or 47% rounded.

Now, a strategy is just a rule telling Bob what to do given target T. The strategy above is (12345) since, on hearing T, Bob opens door T. Another strategy one might consider is this: (12222). Here Alice will place card 1 behind door 1. If Bob hears $T = 1$ he opens door 1. Otherwise he opens door 2.

Thus there are exactly 5^5 strategies (though only their "type" matters. There are only seven types: {{5}, {4, 1}, {3, 2}, {3, 1, 1}, {2, 2, 1}, {2, 1, 1, 1}, {1, 1, 1, 1, 1}}; the last means 5 different numbers, the first means all the same: Bob opens door 1 no matter what T is). In each case, Alice should do what is best to maximize Bob's chances of success. I was VERY surprised when I found that strategy (11345) (whose type is {2, 1, 1, 1}) beats (12345). This optimal strategy is: In short: **Bob opens door *T*, except he opens door 1 if *T* = 2.** The total success count for strategy (11345) is 286, so the probability is $\frac{71.5}{150}$ or $47\frac{2}{3}\%$, or 48% rounded. One can quickly check all seven types of strategies to conclusively prove that this is best possible.

The Theorem above has an analog for the new strategy. Let $f_{11\,345}$ be the count.

Theorem. $f_{11\,345....}(n) =$

$$2 \cdot n! - 2 + T(n) - (n-1)! + (n-1)\,T(n-2) + (n-2)!\,(n-2) - (n-2)! \sum_{i=0}^{n-3} \frac{T(i)}{i}.$$

So $f_{11\,345}(5) = 286$.

I will omit details of the proof since, for the problem at hand where n is just 5, a proof by cases can be given.

Case 12: 12XYZ. 6 permutations. Each permutation here has 1 as a success, contributing 6. The numbers XYZ form a permutation that Alice can apply the basic strategy to, since Bob's moves there are the basic strategy. But $f(3) = 2 \cdot 6 - 1 + T(3) = 14$. So the total count in this case is $6 + 14 = 20$.

Case 21: 21XYZ. Identical to Case 12 since the card behind door 1 is a success. 20 more to the count.

Case 13: 13XYZ. Here 3 is a proxy for 3, 4, or 5, thus introducing a factor of 3. Door 1 contributes 1 for each permutation, so 6 in all; we will now ignore door 1. And over these six permutations 4 is a fixed point twice, as is 5. So in all we have 10 fixed points. Now for each of these permutations Alice's switch can add one more fixed point. So the count is now 16. When there is a transposition that does NOT involve 2, her switch will add 2 not 1. The number of permutations with such a transposition here is 1 (only 13254). So the count in this case is 17, which gets multiplied by 3, for 51.

Case 23: 23XYZ identical to Case 13. Another 51 to the count. So far we have
51 + 51 + 20 + 20 = 142.

Case 3: 3XYZW. Here 3 is a proxy for 3, 4, or 5 introducing a factor of 3. If 4 was used, then the next case would be 4XY1W.

Subcase: 3X1ZW: There are 4 fixed points (since we do not count 2 as a fixed point as it leads to a bad choice) and 3! from Alice's switch, and an extra 6 for the (13) transposition that exists in all cases. So 16, and so 48.

Subcase: 3X2ZW : Identical to preceding case. 48 Total is now 96 + 142 = 238.

Subcase: 3X4ZW. Here 4 is a proxy for 4 or 5, introducing a factor of 2. The count here is 6 for Alice's moves, 2 for the possibility of 5 being a fixed point (we ignore 2 being a fixed point), and 0 for transpositions not involving 2. So 8. Doubling gives 16. Tripling gives 48.

Total number of permutations: $6 + 6 + 3 \cdot 6 + 3 \cdot 6 + 3 \cdot 6 + 3 \cdot 6 + 6 \cdot 6 = 120$. Total count of successes is 238 + 48 = 286, as claimed. □

The formula in the theorem gives the exact answer for the double-door method very quickly. When $n = 52$ the double-door method has about 10^{21} more successes than the basic method.

Basic: **1930528693151818958791918907834650744**1924907159953215222954
(1134…): **1930528693151818958791918907834650744**7768491304547942428300

The gain is 5843584144594727205346 The success probability for the basic method is about: 4.60283%. The probability increase using double-door is about 10^{-38}.

But we can do *much* better for $n = 52$. Consider the multiple double door strategy: (11223344…). After some hard work (that, like so much in math, seems obvious in practice), I have a formula for the count for this strategy. It uses inclusion-exclusion as in the basic case. The formula (for n divisible by 4) is

$$2\,n! - 2^{n/2}\,\frac{n}{2}! + \sum_{j=1}^{\frac{n}{4}} \frac{(2\ 2)^j\left((-1)^{j+1}\ \frac{n}{2}!\right)(n-2\,j)!}{2^j\,j!\left(\frac{n}{2}-2\,j\right)!}$$

Comparing the three methods:

(1234…): **1930528693151818958791918907834650744** 1924907159953215222954
(1134…): **1930528693151818958791918907834650744** 7768491304547942428300
(1122…): **19305** 790553019728990959367482833789315254882161900348781258

The gain to the probability here is substantial: the count increases by 10^{64}. The probability (one just divides by $52\cdot 52!$) increases from 4.60283% to 4.60295%. So naturally one wonders:

Open Problem 1. Is there a better strategy for $n = 52$?

The number of possible strategies is not large: 281589 types for 52 (the number of integer partitions of 52). But it is not clear how to rank each over all of Charlie's $52\cdot 52!$ choices. Perhaps for smaller n one can do this. I have done this up to $n = 10$.

Another open question is this. Simon Plouffe has found empirically that $T(n)$, the number of permutations with a transposition, is $T(n) = n! - \frac{n!\left\lfloor \frac{1}{2} + \frac{\lfloor \frac{n}{2} \rfloor!\, 2^{\lfloor \frac{n}{2} \rfloor}}{\sqrt{e}} \right\rfloor}{2^{\lfloor \frac{n}{2} \rfloor}\lfloor \frac{n}{2} \rfloor!}$. If $n = 2\,k$ is even, this looks better as: $T(n) = n!\left(1 - \frac{\left\lfloor \frac{1}{2} + \frac{k!\,2^k}{\sqrt{e}} \right\rfloor}{2^k\,k!}\right)$. Or we can express this as the probability that a permutation has no transpositions: $\left\lfloor \frac{1}{2} + \frac{k!\,2^k}{\sqrt{e}} \right\rfloor 2^{-k}/k!$. But this is not proved in general.

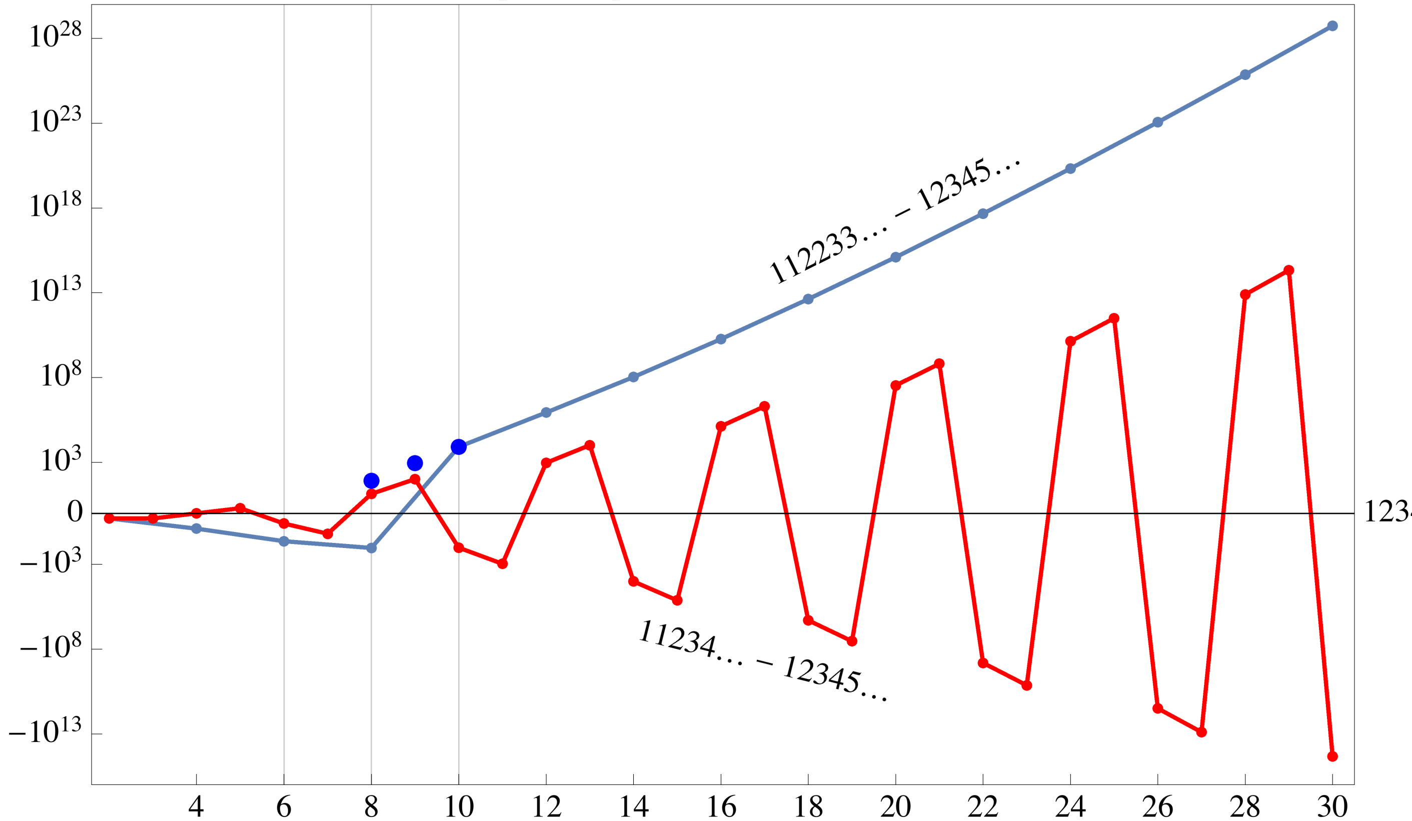
Open problem 2: Prove Plouffe's formula for $T(n)$.

Note that $k!\,2^k/\sqrt{e}$ is not an integer. So the conjectured formula can be written as follows, where $\{\cdot\}$ denotes rounding:

$$\text{Probability(permutation of } 2k \text{ has no transpositions)} \overset{?}{=} \frac{\left\{\frac{k!\,2^k}{\sqrt{e}}\right\}}{k!\,2^k}.$$

Here is a graph of the three methods discussed: 12345…, 11234…, and 11223344…

Two inobvious strategies for the one-flip challenge;
blue dots mark the proved optimal: 11223345 and 112233445 for 8 and 9



Here are the optimal strategies up to $n = 10$, together with the best I know for 11.

$n = 4$: {1, 2, 3, 4} identity
$n = 5$: {1, 1, 2, 3, 4} double-door
$n = 6$: {1, 2, 3, 4, 5, 6} identity
$n = 7$: {1, 2, 3, 4, 5, 6, 7} identity
$n = 8$: {1, 1, 2, 2, 3, 3, 4, 5}
$n = 9$: {1, 1, 2, 2, 3, 3, 4, 4, 5} multiple double-door
$n = 10$: {1, 1, 2, 2, 3, 3, 4, 4, 5, 5} multiple double-door
$n = 11$: {1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6} multiple double-door [not proved best]