Two Geometric Derivations of the Optimal Launch Angle

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We present a couple proofs that the launch angle to strike a given ramp at the farthest distance along the ramp is the angle that splits the difference between the ramp and vertical. If the ramp equals the ground, this yields the classic case that the optimal angle is 45°. Both proofs avoid calculus and trigonometry, except for the most rudimentary; the proof based on the envelope of all trajectories avoids any trigonometry whatsoever. We assume the basic facts about the focus, the directrix, and the reflective property of a parabola. This approach was inspired by Richard Guy's citation of Exercises 39 and 40 on page 30 of Durell's 1927 book "A Concise Geometrical Conics". See: <https://drive.google.com/a/macalester.edu/file/d/0B9uh0VymSVrpNXg1UHZGZUNvbVk/edit>. We also thank Japheth Wood (Bard College) for many helpful comments.

We may assume that the acceleration due to gravity is $g = 1$ and that $v = \sqrt{2}$, where v is the initial speed; we use the easily-derived fact that every trajectory is a vertically-oriented parabola. The ramp angle is A; the origin is O. Consider *any* trajectory at launch angle ϕ from vertical; let $\theta = \frac{\pi}{2} - \phi$, the horizontal launch angle; let $F = (F_x, F_y)$ be the focus of the parabola. Claims 1 to 3 apply to any trajectory; the ramp is not mentioned.

Claim 1. *F* is on the line at angle 2ϕ from vertical.
Proof. By the parabolic reflective property applied to the incoming ray along the negative y-axis, which has incidence angle ϕ (Fig. 1). □

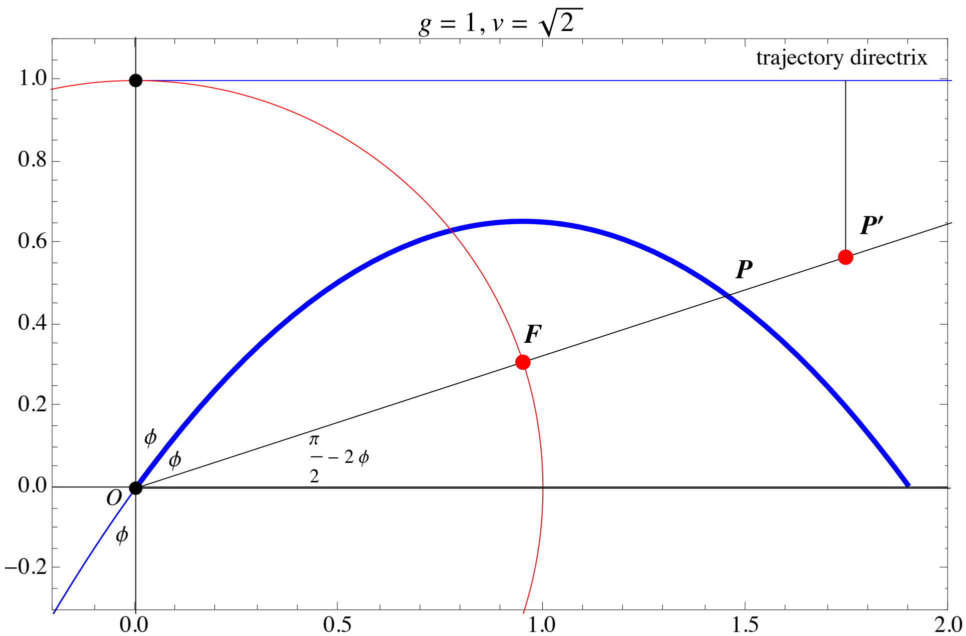

Figure 1. The blue parabola is an arbitrary trajectory; P is the intersection of the parabola with the extend ray OF.

The next two claims give two critical properties of the focus and directrix of a trajectory. An alternate approach follows, which gives first the directrix and then the focus.

Claim 2. *F* is on the unit circle.
Proof. Because $y' = \sqrt{2}\cos(\phi) - t$, the trajectory's high point is reached when $t = \sqrt{2}\cos\phi$ (since that makes $y' = 0$). At that time, x, which is F_x, is $\sqrt{2}\, t \sin\phi = \sqrt{2}\,\sqrt{2}\cos\phi\sin\phi = \sin(2\phi)$. And $F_y = F_x \cot(2\phi) = \sin(2\phi)\cot(2\phi) = \cos(2\phi)$, proving the claim. □

Claim 3. The directrix of the θ-trajectory is the line *d*, given by *y* = 1.
Proof. By Claim 2, $|FO| = 1$ so that is the distance from O to the directrix. The trajectory is vertical and opens downward so the directrix is as claimed. □

We can avoid some of the preceding algebra with an argument that starts with the directrix instead of the focus. The algebra is simpler, but it uses conservation of energy.

Claim 3, Alternate. If a projectile following a parabolic trajectory in a constant gravitational field has total energy E, then the directrix of the parabola is the horizontal line at that height where the potential energy equals E (and kinetic energy is 0). In our specific case above, consideration of the purely vertical trajectory shows that the top is reached when $t = \sqrt{2}$ and its location is $\sqrt{2}\,t - \frac{1}{2}t^2$, or the point $(0, 1)$. Thus the directrix is the line $y = 1$.

Proof. Take the vertex of the parabola to be $(0, 0)$. The velocity there is $(0, a)$ for some real a. The directrix is the line $y = h$, for some $h > 0$, while the focus is $(0, -h)$, below the vertex. Look at the two points on the parabola at height $-h$. There (by the reflective properties) the parabola is at 45° degrees (slope ± 1). So there the velocity is $(a, \pm a)$. The projectile therefore has kinetic energy $\frac{1}{2}\left(2\,a^2\right) = a^2$ at height $-h$, and $\frac{1}{2}a^2$ at height 0. Since an increase in height of h causes a loss of half the kinetic energy, an additional upward move, to height h, will cause the loss of all the kinetic energy. This the kinetic energy is 0 at height h. □

Claim 2, Alternate. The focus of any trajectory lies on the unit circle.
Proof. The parabola contains O and $|FO|$ equals the distance from O to the directrix. So this follows from Claim 3, Alternate. □

Now consider a given ramp at angle A and refer to Figure 1, where now $\frac{\pi}{2} - 2\,\phi = A$, so $\phi = \frac{1}{2}\left(\frac{\pi}{2} - A\right)$.

Claim 4. The farthest point any trajectory reaches on the A-ramp is P.
Proof. Suppose P' is a farther point on the ramp and lies on a trajectory with focus F'. But $|P'F'| \geq |P'F| > |PF| = |Pd| \geq |P'd|$, which contradicts the focus-directrix property of the hypothesized trajectory. If the ramp descends from the origin, a different argument is needed. Look at the right triangle (Fig. 2) with vertical leg δ and hypotenuse H. Then δ is the amount the distance to the directrix increases, while H is, as in the upward case, a lower bound on the amount the distance to the focus increases. Because $\delta < H$, this again contradicts the existence of a parabolic trajectory for P'. □

The vertical leg of the small right
triangle is shorter than its hypotenuse.

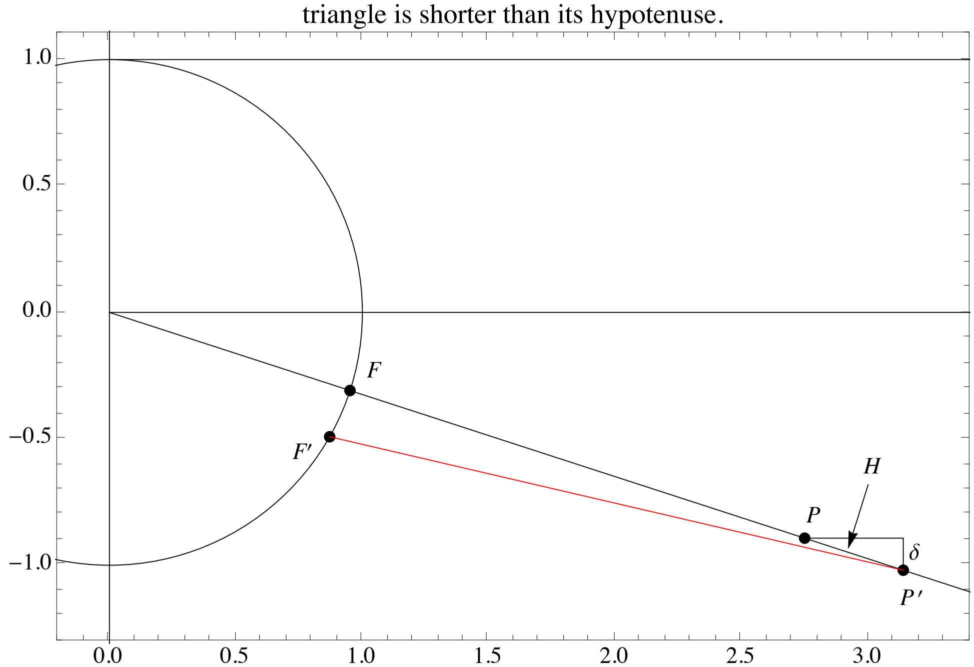

Figure 2. For a downward-sloping ramp at angle $A < 0$, a projectile launched at angle $A + \frac{1}{2}\left(\frac{\pi}{2} - A\right)$ reaches the farthest point, P, on the ramp.

An alternate path to the result makes use of the envelope, an elegant parabola that wraps itself tightly around all the trajectories. Define P_E to be the parabola with focus at the origin and directrix the line $y = 2$ (Fig. 3); P_E is the envelope.

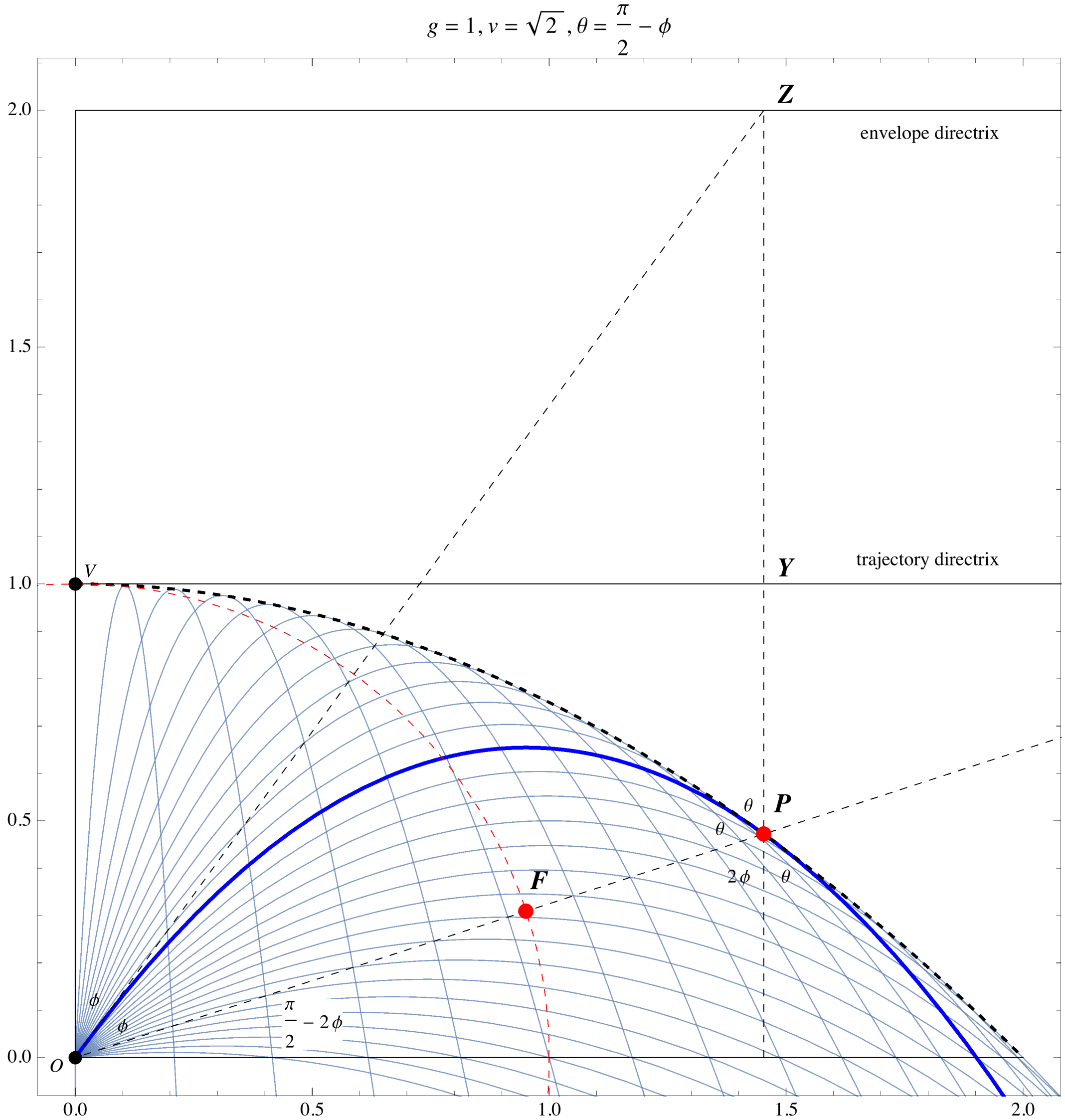

Figure 3. The envelope of all trajectories from the origin is the parabola (dashed) with focus at the origin and directrix the line $y = 2$. The launch direction strikes the envelope directrix directly above the contact point.

Claim 4, Alternate. All trajectory parabolas are tangent to and lie inside P_E. Given a ramp at angle A, the optimal trajectory is the one launched at angle ϕ from vertical, where $\phi = \frac{1}{2}\left(\frac{\pi}{2} - A\right)$. The farthest point reached is where the ramp strikes the envelope.

Proof. Given a trajectory parabola with focus F, let P be its intersection point with the extension of OF. Because $|PY| = |PF|$, Claim 3 implies $|PZ| = |PY| + 1 = |PF| + 1 = |PO|$ (Fig. 3) and so P lies on P_E. And the tangents to the two parabolas at P coincide, because the upward vertical ray into P must reflect to the line OF in either case, as the foci are O and F. So P_E is the envelope of all the trajectories. But we also need to know that all the trajectories lie below P_E. Consider all vertical parabolas that pass through P and have the same slope that P_E has there. Translating coordinates so that P is the origin, all such parabolas have the form $y = a\,x^2 - b\,x$. But this family, as a varies, is just a shear of the family $\{a\,x^2\}$ and is therefore nested. Since the trajectories start at a point under P_E, they all stay completely under P_E. Therefore, for a ramp at angle A, the optimal trajectory is the one launched at angle ϕ from vertical, where $\phi = \frac{1}{2}\left(\frac{\pi}{2} - A\right)$. □

Note that the straight line from the origin at the launch angle strikes P_E's directrix directly above P. The next result follows from either Claim 4, or the alternate.

Corollary. (a) If the ramp is horizontal, then the optimal angle is 45°. (b) The farthest distance reached measured along the ramp $\frac{2}{1+\sin A}$.
Proof. (a) is immediate using $A = 0$. (b) holds because the expression is just the polar coordinate representation of the envelope. To see that, solve the directrix equation in polar form: $r\{\cos A, \sin A\}$ has length r and distance $2 - r\sin A$ from the directrix; solving $2 - r\sin A = r$ yields $r = \frac{2}{1+\sin A}$.

Note that the approach here yields four distinct proofs.

1. Use conservation of energy (alternate claims 3 and 2) to analyze the trajectories and the envelope analysis to finish.
2. Use the parametrization and rudimentary trig (claims 2 and 3) to analyze the trajectories and the envelope analysis to finish.
3. Use conservation of energy to start and the geometric argument (Claim 4) to finish.
4. Use the parametrization and rudimentary trig to start and the geometric argument to finish.