1213 A Sine Coincidence? Or Not? Consider Ceiling[n/π] and Ceiling[1/sin(π/n)], where $n \geq 2$ is an integer. The sequences start as follows
1,2,2,2,2,3,3,3,4,4,4,5,5,5,6,6,6,7,7,7,8,8,8,8,9,9,9,10,...
1,1,2,2,2,3,3,3,4,4,4,5,5,5,6,6,6,7,7,7,8,8,8,8,9,9,9,10,...
They differ when $n = 3$. Are they equal for all larger n?
Source: S.W. original.

Problem 1213 was solved by David Broadhurst, Joseph DeVincentis, Jonathan Lee, Ross Millikan, and Joshua Zucker. Some others realized how the search should go, but did not carry on to find an explicit example.

The smallest n for which the sequences differ is 80143857; one could find this by a brute force search of all n up to 81 million, but there is a much faster way. The main idea is to realize that for this to fail one needs a good rational approximation to π. Here is the approach taken by Jonathan Lee (Trinity College, Cambridge):

Because $\sin\frac{\pi}{n} < \frac{\pi}{n}$, we have $\frac{n}{\pi} < \frac{1}{\sin\frac{\pi}{n}}$. For the ceilings to differ, there is an integer k with $\frac{n}{\pi} \leq k < \frac{1}{\sin\frac{\pi}{n}}$. From this we have $\frac{n}{k} \leq \pi$ and also $\sin\frac{\pi}{n} < \frac{1}{k}$ or $\frac{\pi}{n} < \arcsin\frac{1}{k}$ or $\pi < \frac{n}{k}\left(k \arcsin\frac{1}{k}\right)$. So we have $\frac{n}{k} \leq \pi < \frac{n}{k}\left(k \arcsin\frac{1}{k}\right)$. Then the series for arcsine implies that $1 + \frac{x^2}{6} \leq \frac{1}{x}\arcsin x$. So we seek rationals $\frac{n}{k}$ such that $\frac{n}{k} \leq \pi \leq \frac{n}{k}\left(1 + \frac{1}{6k^2}\right)$.

It is well known that such good rational approximations to π, if they exist, should occur among the continued fraction convergents to π. Indeed one could ask for more (say, k^3 instead of k^2 in the error term). This is the same as asking if μ, the "irrationality measure" of π, is 3 or larger. This is not known. Computations suggest that no such approximations that are better than quadratic exist for π ($\mu(\pi)$ would then be 2), but this is not known; the best known result is that $\mu(\pi) \leq 7.61$. See http://-mathworld.wolfram.com/IrrationalityMeasure.html. An interesting related question (unsolved) is whether the Flint–Hill series $\Sigma_{n=1}^{\infty} n^{-3} \csc^2 n$ converges. An affirmative answer would imply that $\mu(\pi) \leq \frac{5}{2}$. The general belief is that $\mu(\pi) = 2$. It is known that $\mu(e) = 2$.

Returning to our problem, the convergents of π are: 3, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, $\frac{103\,993}{33\,102}$, … .
These of course are derived from the CF $\pi = 3 + \cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\ldots}}}}$, usually written as

$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, \ldots]$.

Now, because of the direction of the inequality sought, we need only consider every second convergent starting from 3:

$$3,\ \frac{333}{106},\ \frac{103\,993}{33\,102},\ \frac{208\,341}{66\,317},\ \frac{833\,719}{265\,381},\ \frac{4\,272\,943}{1\,360\,120},\ \frac{80\,143\,857}{25\,510\,582},\ \frac{245\,850\,922}{78\,256\,779},\ \ldots$$

And we care only about the numerators:

3, 333, 103 993, 208 341, 833 719, 4 272 943, 80 143 857, 245 850 922, 1 068 966 896, 6 167 950 454, 21 053 343 141, 3 587 785 776 203, 8 958 937 768 937, 428 224 593 349 304, 6 134 899 525 417 045, 66 627 445 592 888 887, 2 646 693 125 139 304 345, 265 099 323 460 521 503 743, 1 850 401 877 973 371 917 511, 37 535 589 513 263 342 053 361, …

A quick check shows that 80143857 (numerator of the 12th convergent where we start counting at 0) is the first one that works: for this n we have the following, where of course the nearest integer is the denominator of the convergent.

$$\frac{n}{\pi} = 25\,510\,581.99999999529\ldots < 25\,510\,582.00000000183\ldots = \csc\frac{n}{\pi}.$$

It is curious that the next several potential examples all work too: 80143857 is the numerator of c_{12}. Then the numerators of c_{14}, c_{16}, c_{18}, c_{20}, c_{22}, and c_{24} also work; but c_{26} does not.

Now natural questions are:

1. Does this occur infinitely often?
2. Are there values of n that work that are not numerators of convergents of π?

David Broadhurst found that sometimes integer multiples work. His first example is: 2 numerator(c_{20}), or 42106686282. In this case, the number is also the numerator of the 19th convergent for 2π. This works also with the 2 replaced by 3, 4, 5, or 6. So to get an unsolved replacement for Question 2:

2A. Are there values of n that work that are not multiples of numerators of convergents of π?

[21, 6]
[33, 7]
[79, 9]
[307, 15]
[431, 104]
[28421, 202]
[156381, 306]
[267313, 525]
[453293, 2608]

The last entry means that 2608 times the numerator of the 453293-th convergent to π is a solution. This is the numerator of a convergent to 2608 π, but not necessarily the 453293-th. Here the indexing of the convergents is starting at 1 (as opposed to the more traditional 0).

Jonathan Lee (Cambridge, England) observed that one gets the desired convergent whenever an entry greater than 1 occurs in the continued fraction of π, in an even position (though this does not give all convergents that work). The CF for π is:

[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2,
2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1, 4, 2, 6, 6, 99, 1, 2, 2, 6, …]

Entries in even positions that are not 1 occur in positions

0,12,16,18,20,24,26,28,30,32,34,36,38,42,44,46,48,50,60,68,70,74,76,78,84,86,88,98,100,...

As noted, this is a subset of the convergent indices that work, which are:

0,12,14,16,18,20,22,24,26,28,30,32,34,36,38,40,42,44,46,48,50,58,60,62,66,68,70,74,76,78,84,86,88,96,98,100,...

But it allows for quick checking, and Lee found that of the first 400000 terms of the CF, 234620 meet his condition. So it seems quite likely (though still not proven) that there are infinitely many examples.