Problem 1212. A Rare, if Obtuse, Ratio

Find an obtuse triangle with sides of integer length and having two acute angles in the ratio 7 to 5.

Source: Dick Hess, All-Star Mathlete Puzzles, Sterling Publishing Co., New York, 2009, Problem 60. The book is a lovely collection of unusual puzzles.

Problem 1212. A Rare, if Obtuse, Ratio Find an obtuse triangle with sides of integer length and having two acute angles in the ratio 7 to 5. Source: Dick Hess, *All-Star Mathlete Puzzles*, Sterling Publishing Co., New York, 2009, Problem 60. The book is a lovely collection of unusual puzzles. A simpler version of this problem appears as #129 in *The USSR Olympiad Problem Book*, by Shklarsky, Chentsov, and Yaglom, Freeman, San Francisco, 1962.

SOLUTION

Problem 1212 was solved by David Broadhurst, Joseph DeVincentis, Russ Gordon, Richard Boardman, John Snyder, Walter Taylor, and Franz Pichler.

Problem 1212 was first posed by Dick Hess in the Pi Mu Epsilon Journal and the notes below follow the solution of William Pierce (Pi Mu Epsilon Journal, Problem 971, 11:3 Fall 2000, 159-160). These notes start with an elementary solution to the problem, and then discuss a more sophisticated approach uses Chebyshev polynomials (David Broadhurst, Russ Gordon); there is also an approach using Gaussian integers (Walter Taylor). It is possible that the Chebyshev polynomial approach could lead to a proof of minimality of the smallest solution. Those approaches also allow an easy algorithm to solve the problem for any other ratio in place of 5:7.

Let the sides be a, b, c, opposite angles $A = 5\,\theta$, $B = 7\,\theta$, and $C = \pi - 12\,\theta$. Straightforward trig expansion and the Law of Sines then yields a constant k and polynomials p_i so that:

$$a = k\sin(5\,\theta) = k\sin(\theta)\,p_1(\cos\theta)$$
$$b = k\sin(7\,\theta) = k\sin(\theta)\,p_2(\cos\theta)$$
$$c = k\sin(12\,\theta) = k\sin(\theta)\,p_3(\cos\theta)$$

with the polynomials p_i given by:

$$p_1 = 16\,x^4 - 12\,x^2 + 1$$
$$p_2 = 64\,x^6 - 80\,x^4 + 24\,x^2 - 1$$
$$p_3 = 2048\,x^{11} - 5120\,x^9 + 4608\,x^7 - 1792\,x^5 + 280\,x^3 - 12\,x$$

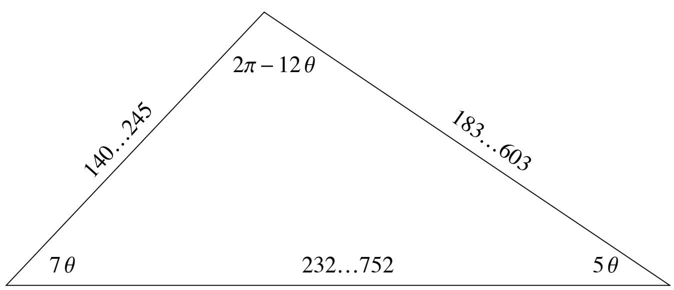
The Cosine Law proves that $\cos\theta$ will be rational. One could just set this to $\frac{r}{s}$, but it is better to assume the denominator is even — $\cos\theta = \frac{r}{2\,s}$ — as this will lead to smaller solutions because of cancellation of powers of 2 with the even coefficients. Let $k = \frac{s^{11}}{\sin\theta}$, which will yield integers. (The original proposer (Dick Hess) and many solvers among you, used the more natural, but less efficient, choice of $\cos\theta = \frac{r}{s}$.) Then we get:

$$a = s^{11}\, p_1\left(\frac{r}{2\,s}\right) = s^{11}\;\frac{r^4 - 3\,r^2\,s^2 + s^4}{s^4} = s^7\left(r^4 - 3\,r^2\,s^2 + s^4\right)$$
$$b = s^{11}\, p_2\left(\frac{r}{2\,s}\right) = s^{11}\;\frac{r^6 - 5\,r^4\,s^2 + 6\,r^2\,s^4 - s^6}{s^6} = s^5\left(r^6 - 5\,r^4\,s^2 + 6\,r^2\,s^4 - s^6\right)$$
$$c = s^{11}\, p_3\left(\frac{r}{2\,s}\right) = s^{11}\;\frac{r^{11} - 10\,r^9\,s^2 + 36\,r^7\,s^4 - 56\,r^5\,s^6 + 35\,r^3\,s^8 - 6\,\mathrm{rs}^{10}}{s^{11}} =$$
$$r\left(r^{10} - 10\,r^8\,s^2 + 36\,r^6\,s^4 - 56\,r^4\,s^6 + 35\,r^2 s^8 - 6\,s^{10}\right)$$

The obtuse angle forces $\theta < \frac{\pi}{24}$, so $\cos\theta < \cos\frac{\pi}{24} = 0.9914\ldots = 1 - \frac{1}{116.889\ldots}$. This means that $2s$ needs to be at least 117, so we take it to be 118 (and this is where one gets a larger solution by taking the denominator to be not 118, but 117, and changing the s^{11} to $(2\,s)^{11}$). Then r is 117 and the formula above give the solution

$a = 140\,737\,857\,915\,018\,789\,245 \sim 1.4 \cdot 10^{20}$
$b = 183\,542\,735\,119\,347\,169\,603 \sim 1.8 \cdot 10^{20}$
$c = 232\,117\,687\,881\,273\,946\,752 \sim 2.3 \cdot 10^{20}$

Without the even denominator trick, one gets solutions near 10^{23}. Here is a picture of the likely smallest solution.



Here is a solution to the given problem by Walter Taylor (Univ. of Colorado); a similar technique (two right triangles) was used by Richard Boardman. Let (a, b, c) be a Pythagorean triangle whose slope is less than tan(7.5°). The smallest such is (17, 144, 145). Let $z = a + b\,i$. Write $z^5 = r + s\,i$, with length t, and $z^7 = u + v\,i$, with length w; all variables are integers. Note that (r, s, t) and (u, v, w) are Pythagorean triples. Assume that r and u are the short sides. So now form the Pythagorean triangles $(u\,r, u\,s, u\,t)$ and $(r\,u, r\,v, r\,w)$. Glue these together along the short sides to get the desired triangle. Doing this for (17, 144, 145) leads to

(372404163668243389715937 5, 281666891054882101164062 5,
501577609454259316429651 2).

This is of size about 10^{24}, which is a bit large, but it satisfies the extra condition that the altitude is an integer (as are the two pieces of the longest side).

David Broadhurst (Open University, UK) (and also Russ Gordon, Whitman College), used Chebyshev polynomials to solve the problem. If $U_n(x)$ is the Chebyshev polynomial of the second kind, then $\frac{\sin((n+1)\theta)}{\sin(n\theta)} = U_n(c) = \sum_{n \geq 2k \geq 0} (-1)^k \binom{n-k}{k} (2c)^{n-2k}$, where $c = \cos\theta$. Let the acute angles be 5θ and 7θ. Then $\theta < \frac{\pi}{24}$, and the cosines of the three angles of the triangle are all rational. The Sine Law then tells us that the sides are in proportion $U_4(c) : U_6(c) : U_{11}(c)$. To get a small solution, set $c = \frac{2n-1}{2n}$ with $n = 59$, the smallest value consistent with $c > \cos\left(\frac{\pi}{24}\right)$. Then the smallest side has length

$$59^{11} U_4\left(\frac{1}{118}\right) = 59^7 \left(59^4 - 3(59 \cdot 117)^2 + 117^4\right) = 59^7 \left(\left(117^2 - 59^2\right)^2 - (59 \cdot 117)^2\right)$$
$$= 140\,737\,857\,915\,018\,789\,245$$

I investigated the general case where (5, 7) is replaced by (m, n); the methods above all work the same way.

I wondered if there is a formula for the simplest case, where $m = 1$ and David Broadhurst showed that a solution is given by $\left\lceil \frac{1}{4\sin^2\left[\frac{\pi}{4n+4}\right]} \right\rceil^n$, where this gives the shortest side of the triangle. Whether this solution, and the ones presented above, are the smallest is still open.