

Stan Wagon POW 1216 - Distances in Sequence

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Introduction

POW 1216: Is there a set of points in the plane such that the set of squared distances between all pairs of points (ignoring duplicates) is the set $\{0, 1, 2, 3, 4\}$?

We show that, up to isometry, there is a unique solution to this problem.

More generally, for $n \in \mathbb{N}$, call a finite subset of the plane a \mathcal{W}_n set if the set of squared distances between all pairs of points in the set (ignoring duplicates) is the set $\{0, 1, 2, \dots, n\}$. We determine all \mathcal{W}_n sets for $n = 1, \dots, 4$ and show that there is no \mathcal{W}_5 set.

Our method is to suppose that \mathcal{S} is a \mathcal{W}_n set including two points A and B , distance \sqrt{k} apart, where $k \in \{1, 2, \dots, n\}$. For convenience, we suppose these points have coordinates $(-\sqrt{k}/2, 0)$ and $(\sqrt{k}/2, 0)$, respectively. With A and B as centres, we then “draw” arcs with radii \sqrt{r} ($r = 1, 2, \dots, n$) and identify the set \mathcal{I}_n of all *intersection points* of these arcs. Then \mathcal{S} must be a subset of $\{A, B\} \cup \mathcal{I}_n$. We then determine which subsets of \mathcal{I}_n (if any), together with A and B , give \mathcal{W}_n sets.

A subset \mathcal{T} of \mathcal{I}_n with at least two elements is said to be *acceptable* if $|XY|^2 \in \{0, 1, 2, \dots, n\}$ for all $X, Y \in \mathcal{T}$. Knowing the acceptable sets, in particular the acceptable pairs, helps to identify \mathcal{W}_n sets.

We initially considered two approaches to the above method: (a) taking $k = n$ for all n , (b) taking $k = 1$ for all n . The latter has the advantage that the intersection points have relatively simple coordinates and the sets \mathcal{I}_n are nested; thus $\mathcal{I}_n \subset \mathcal{I}_{n+1}$. However, in this article we pursue the former which gains by having considerably fewer acceptable pairs when $n = 4$ and 5 .

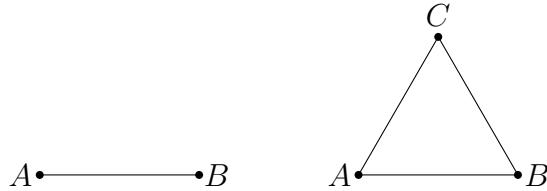
For the notation used in the various cases of (a), together with coordinates of intersection points and lists of acceptable pairs, the reader is directed to the Appendix where this information is gathered together. The Appendix also includes the same information for (b) should any reader wish to try that approach.

The cases $n = 1$ and $n = 2$ are essentially trivial, and even $n = 3$ is quite straightforward, but we include them for the sake of completeness and to illustrate the method with simple examples. Our arguments are highly computational—this may reflect our failure to find inductive methods—and although all calculations have been confirmed “by hand”, the viability of the technique was first checked out with a combination of analogue and digital methods, i.e. ruler and compasses, and elementary programming. The latter (perhaps applied to approach (b)) may well prove to be the only realistic option for applying the method to higher values of n .

When $n = 1, 2, \dots, 5$ and $k = n$

Case $n = 1$. See Figure 1 in the Appendix.

The two points A and B already form a \mathcal{W}_1 set. We try adding the intersection point C (equivalently C^*). Checking distances, we find that $\{A, B, C\}$ (equivalently $\{A, B, C^*\}$) is a \mathcal{W}_1 . There is no acceptable pair to add. Thus:

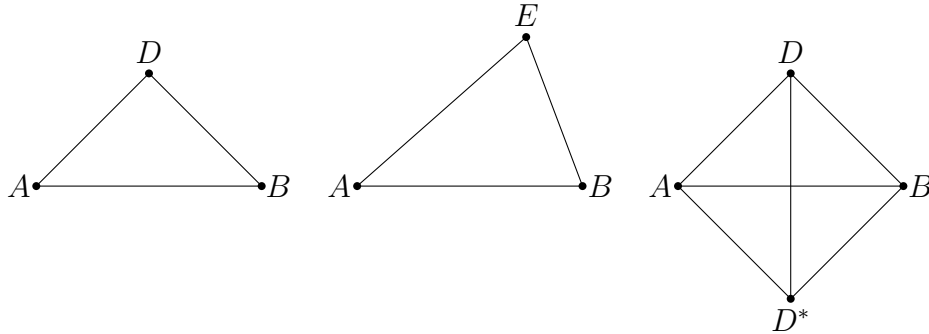


Proposition. The only (isometrically different) \mathcal{W}_1 sets are:

- (1) two points distance 1 apart,
- (2) the vertices of a $(1, 1, 1)$ -triangle.

Case $n = 2$. See Figure 2 in the Appendix.

Here we must add at least one intersection point to A and B . We try C , D and E in turn; each of the other intersection points is symmetrically equivalent to one of these. Checking distances, we find that $\{A, B, D\}$ and $\{A, B, E\}$ are \mathcal{W}_2 sets but $\{A, B, C\}$ is not. To look for larger \mathcal{W}_2 sets we try adding acceptable sets to A and B . In fact, $\{D, D^*\}$ is the only acceptable pair and hence the only acceptable set. Again checking distances, we find that $\{A, B, D, D^*\}$ is a \mathcal{W}_2 set. Thus:



Proposition. The only (isometrically different) \mathcal{W}_2 sets are:

- (1) the vertices of a $(1, 1, \sqrt{2})$ -triangle,
- (2) the vertices of a $(1, \sqrt{2}, \sqrt{2})$ -triangle,
- (3) the vertices of a 1-square.

Case $n = 3$. See Figure 3 in the Appendix.

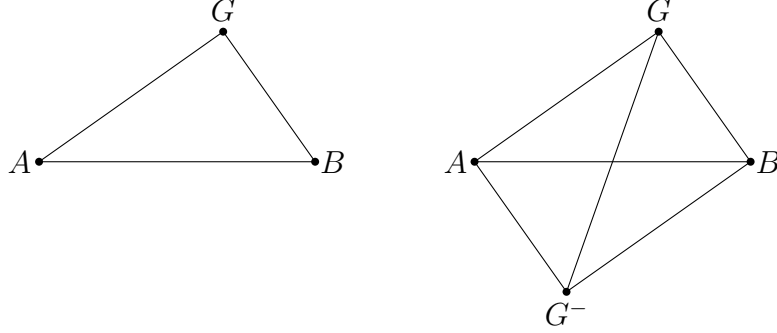
For a three point \mathcal{W}_3 set we must add a point X to A and B such that $|AX|^2 = 1$ and $|BX|^2 = 2$ (or vice versa). This can only be achieved by taking X to be G or a symmetric equivalent (G^* , G' or G^-), and then $\{A, B, G\}$ is a \mathcal{W}_3 set.

For a larger \mathcal{W}_3 set we must add an acceptable set to A and B . The acceptable pairs

are

$$\{C, E\}, \quad \{E, E^*\} \quad \text{and} \quad \{G, G^-\}$$

and the symmetric equivalents $\{C^*, E^*\}$ and $\{G', G^*\}$. Any acceptable triple must be such that each two element subset is an acceptable pair. So there are no acceptable triples and hence no larger acceptable sets. Checking the sets $\{A, B, C, E\}$, $\{A, B, E, E^*\}$ and $\{A, B, G, G^-\}$ we find that only the last is a \mathcal{W}_3 set. Thus:



Proposition. The only (isometrically different) \mathcal{W}_3 sets are:

- (1) the vertices of a $(1, \sqrt{2}, \sqrt{3})$ -triangle,
- (2) the vertices of a $(1, \sqrt{2})$ -rectangle.

Case $n = 4$. See Figure 4 in the Appendix.

Any \mathcal{W}_4 set must have at least four points. So we must add an acceptable set to A and B . The acceptable pairs are

$$\{O, E\}, \{O, J\}, \{C, J\}, \{H, I'\}, \{I, I'\}, \{J, J'\}, \{O, D\}, \{O, I\}, \{H, H^-\},$$

$$\{O, C\}, \{J, J^*\}, \{E, E^*\}, \{H, I^*\}, \{J, J^-\}$$

and symmetric equivalents. The only intersection points in the closed first quadrant which appear in an acceptable pair are O, C, D, E, H, I and J . So any acceptable triple, or one of its symmetric equivalents, must take the form $\mathcal{P} \cup \{X\}$ where \mathcal{P} is an acceptable pair (including all symmetric equivalences) and $X \in \{O, C, D, E, H, I, J\}$. Hence, the acceptable triples are

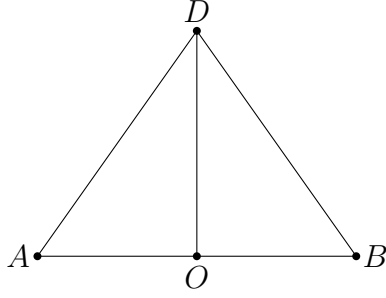
$$\{O, C, J\}, \{O, E, E^*\}, \{O, I, I'\}, \{O, J, J'\}, \{O, J, J^-\},$$

$$\{O, J, J^*\}, \{C, J, J'\}, \{H, H^-, I'\}, \{J, J', J^-\}$$

and symmetric equivalents. Arguing similarly, the only larger acceptable sets are

$$\{O, C, J, J'\}, \{O, J, J', J^-\}, \{J, J', J^-, J^*\}, \{O, J, J', J^-, J^*\}$$

and symmetric equivalents. Checking these with A and B we find that only $\{A, B, O, D\}$ is a \mathcal{W}_4 set. Thus:



Proposition. The only (isometrically different) \mathcal{W}_4 sets are: the vertices of a $(\sqrt{3}, \sqrt{3}, 2)$ -triangle together with the mid point of the longest side.

Case $n = 5$. See Figure 5 in the Appendix.

Here we show that there is no \mathcal{W}_5 set. Suppose there is. Then it must have at least four points. So we must add at least an acceptable pair to A and B . The acceptable pairs are

$$\begin{aligned} & \{G, N'\}_1, \{G, P\}_1, \{H, M'\}_1, \{J, J^-\}_1, \{J, N\}_1, \{J, P'\}_1, \\ & \{G, J'\}_2, \{J, P^*\}_2, \{N, P'\}_2, \{F, F^*\}_3, \{M, M^-\}_3, \{J, N^-\}_4, \\ & \{G, J^*\}_5, \{H, M^*\}_5, \{I, I^-\}_5, \{N, P^*\}_5, \{P, P^-\}_5 \end{aligned}$$

and symmetric equivalents; the subscripts indicate the squared distances between the points.

Only $\{J, N^-\}$ (and its symmetric equivalents) has points which are squared distance 4 apart. So we must either add this pair to A and B , or add (at least) an intersection point X where $|AX|^2 = 4$ (equivalently $|BX|^2 = 4$). We may therefore suppose that our \mathcal{W}_5 set contains $\{A, B, J, N^-\}$ or $\{A, B, X\}$ where X is G, D, H, L or P . Since we have to add at least one point to $\{A, B, X\}$ and no acceptable pair includes D or L we are left to consider (sets containing)

$$\{A, B, J, N^-\}, \quad \{A, B, G\}, \quad \{A, B, H\} \quad \text{or} \quad \{A, B, P\}.$$

We eliminate each of these in turn.

The set $\{A, B, J, N^-\}$ lacks the squared distance 3. Considering all acceptable pairs (including symmetric equivalences), we can only add G^*, P' or P^* . Since none of these, singly or combined, adds a squared distance 3, we cannot extend $\{A, B, J, N^-\}$ to a \mathcal{W}_5 set.

The set $\{A, B, G\}$ lacks the squared distance 3. We can only add N', P, J' or J^* . Since no combination of these adds a squared distance 3, we cannot extend $\{A, B, G\}$ to a \mathcal{W}_5 set.

The set $\{A, B, H\}$ lacks the squared distance 2. We can only add M' or M^* . Since no combination of these adds a squared distance 2, we cannot extend $\{A, B, H\}$ to a \mathcal{W}_5 set.

The set $\{A, B, P\}$ lacks the squared distance 3. We can only add G, J', J^*, N', N^* or P^- . Since no combination of these adds a squared distance 3, we cannot extend $\{A, B, P\}$ to a \mathcal{W}_5 set. Thus:

Proposition. There is no \mathcal{W}_5 set.

Appendix - Intersection Points and Acceptable Pairs

The figures, below, give information about the intersection points and acceptable pairs referred to earlier. Figures 1 to 5 cover, separately, the cases where $n = 1, 2, \dots, 5$ and $|AB| = n$. Figure 6 covers all five cases where $n = 1, 2, \dots, 5$ and $|AB| = 1$. Each figure consists of the following.

(1) A diagram showing A , B and the intersection points. Only the points A and B , and the intersection points in the closed first quadrant are labelled. Other intersection points are identified as follows. With $x, y > 0$, if

$$X = (x, 0), \quad Y = (0, y), \quad Z = (x, y)$$

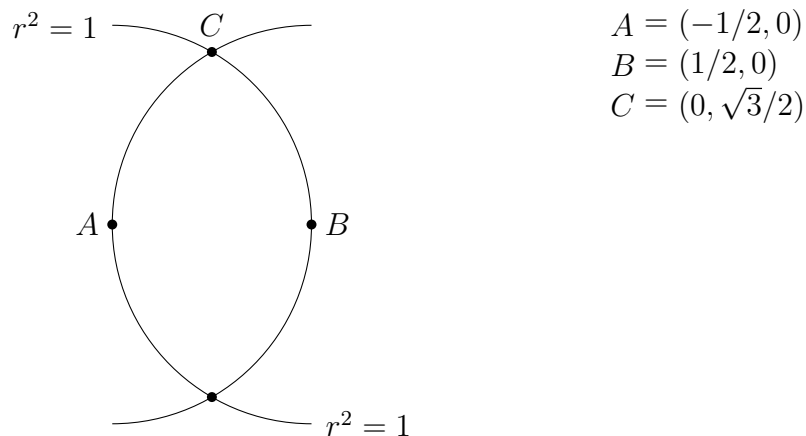
then

$$X' = (-x, 0), \quad Y^* = (0, -y), \quad Z' = (-x, y), \quad Z^- = (-x, -y), \quad Z^* = (x, -y).$$

(2) The coordinates of A and B , and the intersection points in the closed first quadrant. These were determined as follows: given $A = (-t, 0)$, $B = (t, 0)$ and a point (x, y) in the first quadrant distant p from A and q from B , then $x = (t^2 - q^2)/(4t)$ and $y^2 = p^2 - (t+x)^2$.

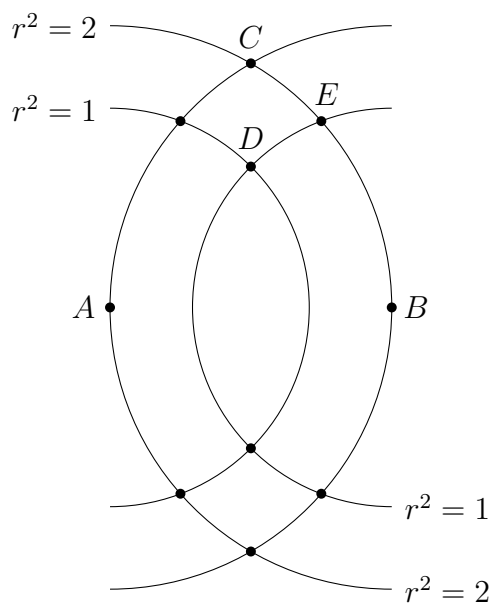
(3) A list of the acceptable pairs of the form $\{U, V\}$, where U or V lies in the closed first quadrant, with just one representative of each “symmetry class” included. So, for example, if $\{J, N^-\}$ is listed then the *symmetric equivalents* $\{J^-, N\}$, $\{J', N^*\}$ and $\{J^*, N'\}$ are also acceptable pairs but are not included in the list. Subscripts give the squared distances between the points; thus $\{J, N^-\}_4$ means that $|JN^-|^2 = 4$.

Figure 1: $n = 1, \quad |AB| = 1$



There is no acceptable pair.

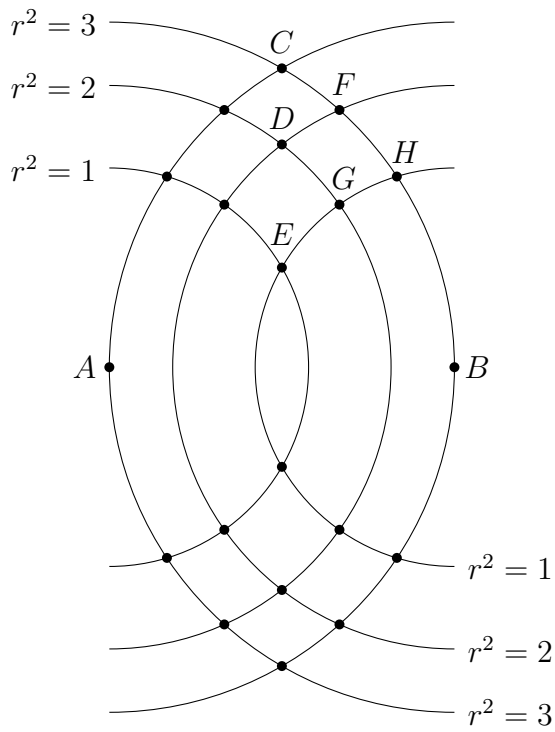
Figure 2: $n = 2$, $|AB| = \sqrt{2}$



$$\begin{aligned}
 A &= (-\sqrt{2}/2, 0) \\
 B &= (\sqrt{2}/2, 0) \\
 C &= (0, \sqrt{6}/2) \\
 D &= (0, \sqrt{2}/2) \\
 E &= (\sqrt{2}/4, \sqrt{14}/4)
 \end{aligned}$$

Acceptable pair: $\{D, D^*\}_2$.

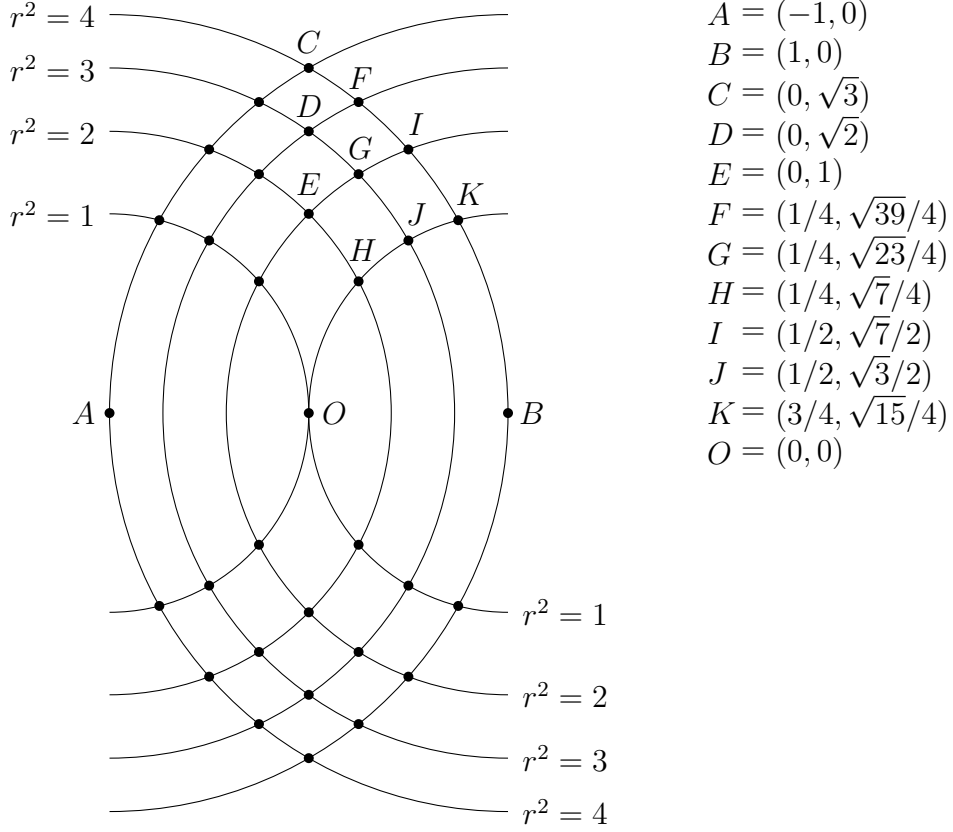
Figure 3: $n = 3$, $|AB| = \sqrt{3}$



$$\begin{aligned}
 A &= (-\sqrt{3}/2, 0) \\
 B &= (\sqrt{3}/2, 0) \\
 C &= (0, 3/2) \\
 D &= (0, \sqrt{5}/2) \\
 E &= (0, 1/2) \\
 F &= (\sqrt{3}/6, \sqrt{15}/3) \\
 G &= (\sqrt{3}/6, \sqrt{6}/3) \\
 H &= (\sqrt{3}/3, \sqrt{33}/6)
 \end{aligned}$$

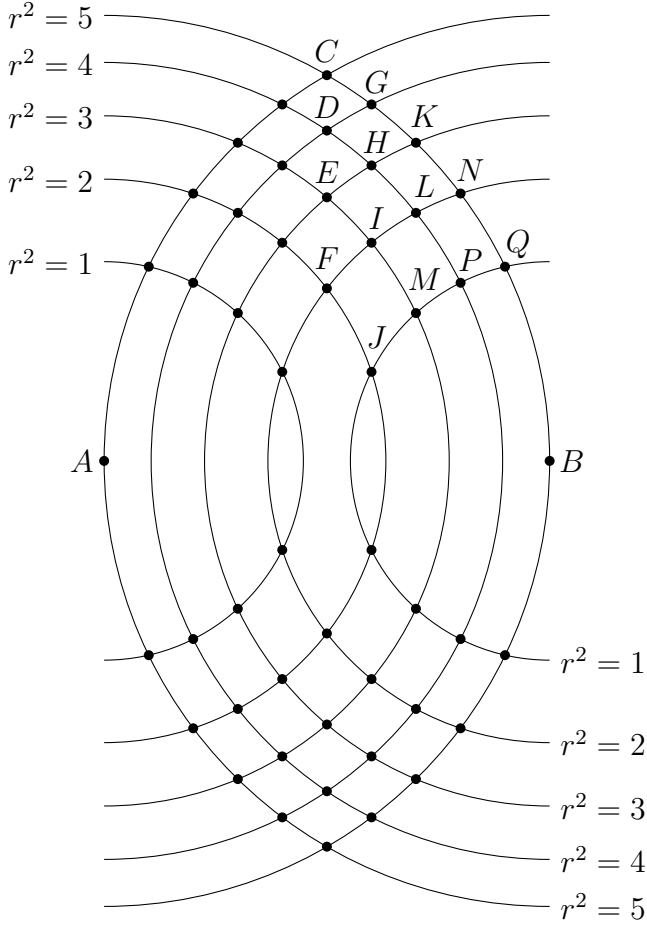
Acceptable pairs: $\{C, E\}_1, \{E, E^*\}_1; \{G, G^-\}_3$.

Figure 4: $n = 4$, $|AB| = 2$



Acceptable pairs: $\{O, E\}_1, \{O, J\}_1, \{C, J\}_1, \{H, I\}_1, \{I, I'\}_1, \{J, J'\}_1$;
 $\{O, D\}_2, \{O, I\}_2, \{H, H^-\}_2, \{O, C\}_3, \{J, J^*\}_3$;
 $\{E, E^*\}_4, \{H, I^*\}_4, \{J, J^-\}_4$.

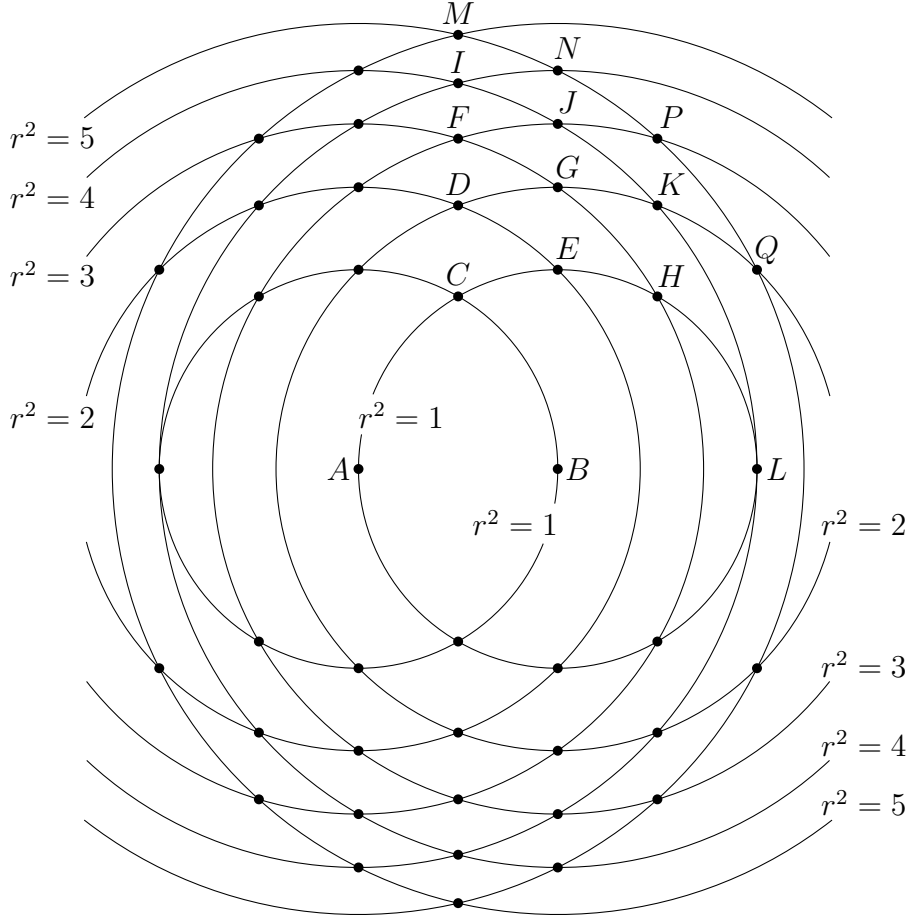
Figure 5: $n = 5$, $|AB| = \sqrt{5}$



- $A = (-\sqrt{5}/2, 0)$
- $B = (\sqrt{5}/2, 0)$
- $C = (0, \sqrt{15}/2)$
- $D = (0, \sqrt{11}/2)$
- $E = (0, \sqrt{7}/2)$
- $F = (0, \sqrt{3}/2)$
- $G = (\sqrt{5}/10, 4\sqrt{5}/5)$
- $H = (\sqrt{5}/10, \sqrt{55}/5)$
- $I = (\sqrt{5}/10, \sqrt{30}/5)$
- $J = (\sqrt{5}/10, \sqrt{5}/5)$
- $K = (\sqrt{5}/5, \sqrt{255}/10)$
- $L = (\sqrt{5}/5, \sqrt{155}/10)$
- $M = (\sqrt{5}/5, \sqrt{55}/10)$
- $N = (3\sqrt{5}/10, 3\sqrt{5}/5)$
- $P = (3\sqrt{5}/10, 2\sqrt{5}/5)$
- $Q = (2\sqrt{5}/5, \sqrt{95}/10)$

Acceptable pairs: $\{G, N'\}_1, \{G, P\}_1, \{H, M'\}_1, \{J, J^-\}_1, \{J, N\}_1, \{J, P'\}_1;$
 $\{G, J'\}_2, \{J, P^*\}_2, \{N, P'\}_2; \{F, F^*\}_3, \{M, M^-\}_3;$
 $\{J, N^-\}_4; \{G, J^*\}_5, \{H, M^*\}_5, \{I, I^-\}_5, \{N, P^*\}_5, \{P, P^-\}_5.$

Figure 6: $n = 1, 2, \dots, 5$, $|AB| = 1$



$$\begin{array}{llll}
 A = (-1/2, 0) & E = (1/2, 1) & I = (0, \sqrt{15}/2) & M = (0, \sqrt{19}/2) \\
 B = (1/2, 0) & F = (0, \sqrt{1}/2) & J = (1/2, \sqrt{3}) & N = (1/2, 2) \\
 C = (0, \sqrt{3}/2) & G = (1/2, \sqrt{2}) & K = (1, \sqrt{7}/2) & P = (1, \sqrt{11}/2) \\
 D = (0, \sqrt{7}/2) & H = (1, \sqrt{3}/2) & L = (3/2, 0) & Q = (3/2, \sqrt{95}/10)
 \end{array}$$

No acceptable pair when $n = 1$.

Acceptable pairs when $n = 2$: $\{E, E'\}_1$.

Additional pairs when $n = 3$: $\{G, G'\}_1, \{J, J'\}_1, \{C, C^*\}_3, \{H, H^*\}_3$.

Additional pairs when $n = 4$: $\{C, H\}_1, \{C, J\}_1, \{D, K\}_1, \{H, J\}_1,$
 $\{H, L\}_1, \{E, L\}_2, \{K, L\}_2, \{C, L\}_3, \{G, L\}_3, \{H, J'\}_3, \{C, H^*\}_4,$
 $\{D, L\}_4, \{E, E^*\}_4, \{H, H'\}_4, \{J, L\}_4, \{K, K'\}_4.$

Additional pairs when $n = 5$: $\{E, N\}_1, \{E, Q\}_1, \{F, P\}_1, \{L, Q\}_1,$
 $\{N, N'\}_1; \{E, N'\}_2, \{N, Q\}_2; \{L, P\}_3; \{E, Q'\}_4; \{P, P'\}_4, \{Q, Q^*\}_4,$
 $\{E, E^-\}_5; \{E, L'\}_5, \{E, Q^*\}_5, \{F, L\}_5, \{L, N\}_5, \{N, Q'\}_5.$

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