An Elementary Proof of the Uniform Distribution Theorem
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Fix an interval $I = [c, d) \subseteq [0, 1)$, and let L be the length of the interval: $L = d - c > 0$.

For any $a \in [0, 1)$ and any irrational $\xi > 0$, let $x_n(a, \xi) = a + n\xi \bmod 1$. For $N \in \mathbb{Z}^+$, let $[N] = \{1, 2, \ldots, N\}$, and let

$$s_N(a, \xi) = |\{n \in [N] : x_n(a, \xi) \in I\}|,$$
$$f_N(a, \xi) = \frac{s_N(a, \xi)}{N}.$$

We want to prove: $\lim_{N\to\infty} f_N(a, \xi) = L$.

We start with a couple of lemmas:

Lemma 1. *Suppose that $x < y$. Then*

$$y - x - 1 < |[x, y) \cap \mathbb{Z}| < y - x + 1.$$

Proof. It is not hard to see that $|[x, y) \cap \mathbb{Z}| = \lceil y \rceil - \lceil x \rceil$. The conclusion now follows from the following facts:

$$\lceil y \rceil - \lceil x \rceil = y - x + (\lceil y \rceil - y) - (\lceil x \rceil - x), \qquad 0 \le \lceil x \rceil - x, \lceil y \rceil - y < 1. \qquad \square$$

Lemma 2. *For sufficiently large N,*

$$L - 2\xi < f_N(a, \xi) < L + 2\xi.$$

Proof. Since a and ξ will be fixed throughout the proof of the lemma, we write x_n, s_N, and f_N for $x_n(a, \xi)$, $s_N(a, \xi)$, and $f_N(a, \xi)$.

Notice that for any positive integer n, $x_n \in I$ if and only if there is some integer $k \ge 0$ such that

$$k + c \le a + n\xi < k + d.$$

Let

$$B_k = \{n \in \mathbb{Z}^+ : k + c \le a + n\xi < k + d\}.$$

Then it is easy to see that the sets B_k form a partition of $\{n \in \mathbb{Z}^+ : x_n \in I\}$. The definition of B_k is equivalent to

$$B_k = \left\{ n \in \mathbb{Z}^+ : \frac{k + c - a}{\xi} \le n < \frac{k + d - a}{\xi} \right\} = \left[\frac{k + c - a}{\xi}, \frac{k + d - a}{\xi} \right) \cap \mathbb{Z}^+.$$

Since $a \in [0, 1)$, if $k \ge 1$ then $(k + c - a)/\xi > 0$, so we can apply Lemma 1 to conclude that

$$\frac{k + d - a}{\xi} - \frac{k + c - a}{\xi} - 1 < |B_k| < \frac{k + d - a}{\xi} - \frac{k + c - a}{\xi} + 1.$$

Since $d - c = L$ this simplifies to

$$\frac{L}{\xi} - 1 < |B_k| < \frac{L}{\xi} + 1.$$

For $k = 0$, the restriction that elements of B_0 must be *positive* integers may reduce the number of elements, so all we can say is that

$$|B_0| < \frac{L}{\xi} + 1.$$

Now consider any $N \in \mathbb{Z}^+$, and let $K = \lfloor a + N\xi \rfloor$. We will assume that N is large enough that $K \geq 2$. Then

$$\{n \in [N] : x_n \in I\} = \bigcup_{k=0}^{K-1} B_k \cup (B_K \cap [N]).$$

Using the bounds on the sizes of the sets B_k, we conclude that

$$(K-1)\left(\frac{L}{\xi} - 1\right) < s_N < (K+1)\left(\frac{L}{\xi} + 1\right)$$

By the definition of K, we have

$$K \leq a + N\xi < K + 1,$$

so

$$\frac{K-a}{\xi} \leq N < \frac{K+1-a}{\xi}.$$

Since $a \in [0, 1)$, it follows that

$$\frac{K-1}{\xi} < N < \frac{K+1}{\xi}.$$

Putting together our bounds on s_N and N, we have

$$\frac{(K-1)(L/\xi - 1)}{(K+1)/\xi} < \frac{s_N}{N} < \frac{(K+1)(L/\xi + 1)}{(K-1)/\xi},$$

which simplifies to

$$\frac{K-1}{K+1} \cdot (L - \xi) < f_N < \frac{K+1}{K-1} \cdot (L + \xi).$$

It is clear that if K is large enough, then we will have

$$\frac{K-1}{K+1} \cdot (L - \xi) > L - 2\xi, \qquad \frac{K+1}{K-1} \cdot (L + \xi) < L + 2\xi,$$

and therefore

$$L - 2\xi < f_N < L + 2\xi.$$

And by choosing N large enough, we can ensure that K is large enough to get this conclusion. This proves the lemma. □

Now we're ready to prove that $\lim_{N\to\infty} f_N(a,\xi) = L$. Suppose $\epsilon > 0$. Using the density of $(n\xi \bmod 1)$ in $[0,1)$, choose some M such that $0 < (M\xi \bmod 1) < \epsilon/2$. Let $\bar{\xi} = M\xi \bmod 1$. Now we break the sequence $(x_n(a,\xi))$ up into M subsequences, as follows:

$$\begin{aligned}
&x_1(a,\xi), x_{1+M}(a,\xi), x_{1+2M}(a,\xi), \ldots \\
&x_2(a,\xi), x_{2+M}(a,\xi), x_{2+2M}(a,\xi), \ldots \\
&\quad\vdots \\
&x_M(a,\xi), x_{2M}(a,\xi), x_{3M}(a,\xi), \ldots .
\end{aligned} \tag{1}$$

Notice that for any $n \in \mathbb{Z}^+$,

$$x_{1+(n-1)M}(a,\xi) = a + (1+(n-1)M)\xi \bmod 1 = (a + \xi - M\xi) + n\bar{\xi} \bmod 1.$$

If we let $a_1 = a + \xi - M\xi \bmod 1$, then this means that

$$x_{1+(n-1)M}(a,\xi) = x_n(a_1,\bar{\xi}).$$

In other words, the first row in (1) is the sequence $(x_n(a_1,\bar{\xi}))$. Similarly, we can define numbers $a_2,\ldots,a_M \in [0,1)$ so that for $1 \le k \le M$, row k of (1) is $(x_n(a_k,\bar{\xi}))$.

Now let $N \ge M$ be any integer. For $1 \le k \le M$, let $N_k = \lfloor (N-k)/M \rfloor + 1$. Then the first N terms of $(x_n(a,\xi))$ consist of the first N_k terms of $(x_n(a_k,\bar{\xi}))$, for $1 \le k \le M$. Therefore

$$s_N(a,\xi) = \sum_{k=1}^{M} s_{N_k}(a_k,\bar{\xi}),$$

so

$$f_N(a,\xi) = \frac{\sum_{k=1}^{M} s_{N_k}(a_k,\bar{\xi})}{N} = \sum_{k=1}^{M} \frac{N_k}{N} \cdot \frac{s_{N_k}(a_k,\bar{\xi})}{N_k} = \sum_{k=1}^{M} \frac{N_k}{N} \cdot f_{N_k}(a_k,\bar{\xi}).$$

In other words, $f_N(a,\xi)$ is a weighted average of the numbers $f_{N_k}(a_k,\bar{\xi})$.

Now we apply Lemma 2 to the sequences $(x_n(a_k,\bar{\xi}))$ to conclude that if N_k is sufficiently large, then $L - 2\bar{\xi} < f_{N_k}(a_k,\bar{\xi}) < L + 2\bar{\xi}$. Since $\bar{\xi} < \epsilon/2$, this means that $L - \epsilon < f_{N_k}(a_k,\bar{\xi}) < L + \epsilon$. By choosing N sufficiently large, we can ensure that his holds for all k from 1 to M, and therefore $L - \epsilon < f_N(a,\xi) < L + \epsilon$, as required.