The Locker Problem

Bruce Torrence and Stan Wagon

The locker problem appears frequently in both the secondary and university curriculum [2, 3]. In November 2005, it appeared on National Public Radio's *Car Talk* as a "Puzzler", and so attained even wider circulation [1]. The problem goes like this: A school corridor is lined with 1000 lockers, all closed. There are 1000 students who are sent down the hall in turn according to the following rules. The first student opens every locker. The second student closes every second locker, beginning with the second. The third student changes the state of every third locker, beginning with the third, closing it if it is open and opening it if it is shut. This continues, with the nth student changing the state of every nth locker, until all the students have walked the hallway. The problem is: Which lockers remain open after all the students have marched?

The answer is well known: The lockers whose numbers are perfect squares remain open, as only the squares have an odd number of divisors [4]. We note that this is true whether the corridor contains 1000 or any other number of lockers.

In this note we present some simple techniques for dealing with an extension of this problem. At the outset we wish to extend our thanks to Joe Buhler for lending his attention and sharing his insights.

If we agree that the students are numbered, and that when sent marching, student n will change the state of every nth locker beginning with locker n, then it is known that we can leave *any* collection of lockers open by dispatching precisely the right subset of students [4]. This leads to some interesting problems. For instance, we know that sending all the students leaves the square lockers open. Which subset of students must be sent to leave precisely the cube lockers open? How about the fourth powers? We'll show shortly that there is a simple solution to these questions. In the meantime, we state in full generality the *extended locker problem*: Given a subset of the lockers, which students should be dispatched to keep those lockers open? Conversely, given a subset of the students, which lockers will be left open after they march?

We note that there are several problems interspersed throughout the remainder of this discourse. The impatient reader may wish to attempt these without reading the more general results that fill the space between them. While success is certainly possible, the general results provide a means for tackling most of the specific questions with greater efficiency.

Problem 1. Show that there can be no two distinct sets of students who will leave open the same set of lockers. Hint: Given two sets of students, consider the locker whose number is the smallest where the sets differ.

Returning to our main topic, it is shown in [4] that either version of the extended problem amounts to solving an $m \times m$ nonsingular system of linear equations modulo 2, where m is the total number of lockers in the corridor. While the system has a unique solution for any subset of the lockers (or any subset of the students), finding it this way is tedious, to say the least.

For the remainder of this discourse we will assume that both the number of lockers and the number of students are infinite. The results in Theorems 1 – 3 hold for the case of any finite number m of lockers and students; simply ignore in any of the sets we discuss any numbers exceeding m. We adhere to the convention that the set $\mathbb{N}$ of natural numbers does *not* include 0, so that it corresponds precisely to the complete student and locker sets.

For certain special subsets of the lockers there is a much more elegant way to approach the problem than by solving a large linear system. Toward this end, define the *signature* of a natural number to be the set of all positive exponents appearing in that number's prime factorization. For our purposes it will be convenient to write members of the signature of a number in ascending order. For example, $12 = 2^2 3^1$ and has signature $\{1, 2\}$; $15 = 3^1 5^1$ and has signature $\{1\}$. Note that the signature of 1 is the empty set, $\emptyset$.

It is evident that the squarefree numbers are precisely the numbers whose signature is contained in the set $\{1\}$. The squares are those numbers whose signature is contained in the set $\{2, 4, 6, \ldots\}$, and the cubes are those numbers whose signature is contained in the set $\{3, 6, 9, \ldots\}$. In general, given a subset A of the natural numbers, we let $\sigma(A)$ denote the set of all numbers whose signature is contained in A. With this notation, we have that the set of squarefree numbers is $\sigma(\{1\})$, the set of squares is $\sigma(\{2, 4, 6, \ldots\})$, the set of cubes is $\sigma(\{3, 6, 9, \ldots\})$, and so on. Of course many sets of numbers are not σ of anything. For example, the powers of 2 (or of any specific prime) are not of the form $\sigma(A)$ for any A.

But suppose the set of students is of the form $\sigma(A)$ for some subset A of the natural numbers. Is there an elegant way to characterize the set of lockers that will be left open by these students? Indeed there is. We need one more definition in order to state the result. Given a set A of natural numbers, let $e(A)$ be all natural numbers that are greater than or equal to an *even* number (including 0) of the elements of A. For example, if $A = \{3, 6, 9, \ldots\}$, then $e(A) = \{1, 2, 6, 7, 8, 12, 13, 14, 18, 19, 20, \ldots\} = \{n \in \mathbb{N} : n \equiv 0, 1, 2 \pmod 6\}$.

Theorem 1. Let $A \subseteq \mathbb{N}$. If students $\sigma(A)$ are dispatched then lockers $\sigma(e(A))$ remain open.

Proof. Locker m will remain open if and only if an odd number of students touch it; that is, if m has an odd number of divisors among the numbers in the student set. Suppose that the students $\sigma(A)$ are sent marching, and suppose that locker m has prime factorization $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Note that the divisors of $p_i^{\alpha_i}$ in $\sigma(A)$ are all numbers of the form p_i^{γ}, where $\gamma \in A$ and $1 \leq \gamma \leq \alpha_i$, together with the number $1 = p^0$ (as $1 \in \sigma(A)$ for all sets A). Now $m \in \sigma(e(A))$ if and only if each $\alpha_i \in e(A)$, that is, if each α_i is greater than or equal to $2a_i$ members of A, for some $a_i \geq 0$. If this is the case, then the number of divisors of m among the students who will march is $(2a_1 + 1)(2a_2 + 1) \cdots (2a_n + 1)$, an odd number. Hence locker m remains open. To complete the proof, suppose that some $\alpha_i \notin e(A)$, say α_i is greater than or equal to $2a_i - 1$ members of A. Then the number of divisors of m among the students who will march contains the factor $2a_i - 1 + 1 = 2a_i$, making it even. So locker m will be closed. $\square$

We note that the proof of this result is a straightforward generalization of the standard formal proof of the solution to the original locker problem (see, e.g., [4]). This result makes easy work of several interesting problems (among them the original).

Problem 2. If all the students are sent down the hall, which lockers remain open?

Problem 3. If the squares are sent down the hall, which lockers remain open?

Problem 4. If the nth powers are sent down the hall, which lockers remain open?

Problem 5. If one wishes to keep only locker number 1 open, which students should be sent marching?

The last problem leads to a natural question: Is there an inverse operation to the e function? It is not difficult to see that there is. For a subset A of the natural numbers, let $A + 1$ be the set $\{1\} \cup \{n + 1 : n \in A\}$, and define $f(A)$ to be the symmetric difference of A and $A + 1$, that is, members of the union $A \cup (A + 1)$ that are not common to both.

Problem 6. Show that e and f are inverse operations. That is, show that $e(f(A)) = f(e(A)) = A$ for any set A of natural numbers.

Problem 6 together with Theorem 1 establish the following (which makes Problem 5 a snap):

Theorem 2. Let $A \subseteq \mathbb{N}$. If lockers $\sigma(A)$ are to remain open, students $\sigma(f(A))$ must be dispatched.

Corollary. The set of marching students is in the image of σ if and only if the set of lockers left open is in the image of σ.

Theorem 2 makes light work of these problems:

Problem 7. If one wishes to keep only the cube lockers open, which students should be sent marching?

Problem 8. If one wishes to keep only the nth powers open, which students should be sent marching?

We see now that the extended locker problem is easily solved for any student or locker set that is determined completely by a signature-containing set A; that is, for sets of students or lockers of the form $\sigma(A)$ for some A. However, this is absolutely no help in cases where the student or locker set is not of this form. A different technique can often be brought to bear for such cases.

Problems 5 and 8 provide us with a necessary insight. Note that the answer to Problem 5 can be stated: To keep open only locker number 1, send only the squarefree students. Likewise, the answer to Problem 8 can be stated: To keep open only the lockers that are nth powers, send all students whose number is the product of a squarefree number with an nth power. The squarefree numbers clearly play a crucial role. We let S denote the set of squarefree numbers; $S = \{1, 2, 3, 5, 6, 7, 10, 11, 13, 14, \ldots\}$. And for any natural number m, we let $mS = \{ms : s \in S\}$.

Problem 9. If one wishes to keep only locker m open, show that one should dispatch students mS. Hint: See Problem 5.

Theorem 3. Let $L \subseteq \mathbb{N}$ be the collection of lockers to be kept open. Then student n should be included in the set of marching students if and only if $n \in lS$ for an *odd* number of members $l \in L$.

Proof. Let $L = \{l_1, l_2, \ldots\}$ with $l_1 < l_2 < \cdots$. Note that L may be finite or infinite. One way to keep exactly the lockers in L open is as follows: first send students l_1S. After they have marched only locker l_1 is open. Then send students l_2S. Note that none of the students in l_2S will touch any of the first $l_2 - 1$ lockers. Since $l_1 < l_2$, after this second cadre of students has marched, only lockers l_1 and l_2 will be open. If one were to continue in the fashion, precisely the lockers in L would be open. Now let $n \in \mathbb{N}$. Then $n \in lS$ for only finitely many $l \in L$. Suppose n is an element of precisely k of the sets lS, for $l \in L$. If k is even, then in the above scenario student n will have marched an even number of times. This has the same effect as student n not marching at all. If k is odd, then student n will have marched an odd number of times, and this has the same effect as student n marching just once. $\square$

As several people pointed out to us, one can formulate Theorem 3 quite naturally in terms of Möbius inversion. But we have chosen here to present a totally elementary approach. With Theorem 3 in hand, several other cases of the extended locker problem are within reach.

Problem 10. Let p be prime. If one wishes to keep open only those lockers whose numbers are powers of p, which students must be dispatched? Try this both with $1 = p^0$ included in the locker set, and with it not included.

Problem 11. Which students must be dispatched to keep only the prime lockers open?

As a final observation, we note that there can be no nonempty subset A of the students that keeps precisely the lockers A open when the hallway is infinite.

Theorem 4. Let $A \subseteq \mathbb{N}$ be any nonempty subset of students. Then the set of lockers left open by these students is not A.

Proof. Let $A = \{n_1, n_2, \ldots\}$ with $n_1 < n_2 < \cdots$. Consider locker $2n_1$. The only proper divisor of $2n_1$ in A is n_1. If $2n_1 \in A$ and the students in A march, then only students n_1 and $2n_1$ will touch locker $2n_1$, and so it will be closed. Conversely, if $2n_1 \notin A$, then only student n_1 will touch locker $2n_1$, and so it will remain open. In either case, the student set does not match the locker set. $\square$

Problem 12. The above proof fails in a corridor with a finite number of lockers. Show that there is a set A in any finite corridor where students A will leave open lockers A. If there are more than two lockers in the hallway, there will be many such sets.

We leave the reader with one final problem. For any subset A of the natural numbers, denote by A^2 the set $\{n^2 : n \in A\}$.

Problem 13. The solution to the original locker problem shows there is a set A with the property that when students A march, lockers A^2 remain open (just take $A = \mathbb{N}$). Find a nonempty set A so that when students A^2 march, lockers A remain open.

References

[1] Car Talk (weekly radio program), Nov. 14, 2005, `http://cartalk.com/content/puzzler/transcripts/200546/`.

[2] J. Kaplan and M. Moskowitz, *Mathematics Problem Solving*, Triumph Learning, New York, 2000, p. 6.

[3] R. Leibowitz, "Writing Discretely", from J.G. Rosenstein, ed., *Discrete Mathematics in the Schools*, American Mathematical Society, DIMACS series, Vol. 36, 1997, p. 85.

[4] B. Torrence, "Extending the locker problem", *Mathematica in Education and Research*, 11:1, Jan. 2006, pp. 83–95.

Bruce Torrence
Department of Mathematics
Randolph-Macon College
Ashland, VA 23005
USA
btorrenc@rmc.edu

Stan Wagon
Department of Mathematics
Macalester College
St. Paul, MN 55105
USA
wagon@macalester.edu