Notes on Edge Coloring Queen Graphs

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Solution to King Graph Problem. It is easy to 8-color the edges. The set of edges receiving one color is a matching (a set of disjoint edges). It is easy to find two matchings that cover all the horizontal edges. And the same is true for the vertical edges. And the same for the up-and-right diagonals, and again for the up-and-left diagonals. That's eight matchings and all edges are covered.

Variation suggested by Andrew Thibodeau: What happens for the king graph $KT_{m,n}$ on a toroidal board? Witold Jarnicki has shown that when $3 \le m \le n$, it is 8-colorable in all cases except m and n both odd, in which case $KT_{m,n}$ is not 8-edge-colorable (but is 9-edge colorable by Vizing's theorem). The both-odd case with $m = 3$ required an argument separate from the $m \ge 5$ case. The case $KT_{1,n}$ reduces to just a simple cycle, for which the behavior is easy to determine.

Queen Graphs: Most, but not all, are optimally edge-colorable (Class One)

Notation: Let $Q_{m,n}$, where $m \le n$, denote the graph of queen moves on an $m \times n$ board.

Easy computations: The maximum degree of $Q_{m,n}$ is $3m + n - 5$ if $m = n$ and n is even, $3m + n - 4$ otherwise. The number of edges of $Q_{m,n}$ is
$\frac{1}{2} m(m-1) + \frac{1}{2} n(n-1) + 2\left((n-m+1)\frac{1}{2} m(m-1) + 2\sum_{i=2}^{m-1} \frac{(m-i+1)(m-i)}{2}\right) =$.
$\frac{1}{6} m\left(2 - 2m^2 - 12n + 9mn + 3n^2\right)$

Background: Vizing's Theorem says that every graph has edge-chromatic number d or $d + 1$, where d is the maximum degree. The former are called Class 1, the latter Class 2.

Reference: http://en.wikipedia.org/wiki/Edge_coloring.

Based on extensive computations that always produced class-1 colorings, I dared to conjecture that all the queen graphs were class 1. That turns out to be false. It is easy to compute the maximum degree in the queen graphs: Currently the cases we do not understand (except in some special cases) are those $Q_{m,n}$ where m and n are odd and $m + 2 \le n \le \frac{1}{3}\left(2m^3 - 11m + 12\right)$.

Thanks to Joseph and Witold, we have the following.

Theorem (Joseph DeVincentis): $Q_{m,n}$ is class 1 if m and n are not both odd.
Theorem (Witold Jarnicki): $Q_{m,m}$, with m odd is of class 1; $Q_{m,n}$ with m and n both odd and

$n \geq \frac{1}{3}(2m^3 - 11m + 18)$ is class 2. For example $Q_{3,n}$ is class 2 when n÷13 and

Theorem: The following are class 1 by integer-linear programming: $Q_{3,5}$, $Q_{3,7}$, $Q_{3,9}$, $Q_{3,11}$, $Q_{5,7}$, $Q_{5,9}$, $Q_{5,11}$, $Q_{5,13}$. Some of these took an hour. We used both *Mathematica* and *Sage*.

Unresolved: $Q_{5,15}$, ..., $Q_{5,69}$, and similar for other $Q_{m,n}$ where m and n are odd, $m \geq 11$, and $m + 2 \leq n \leq \frac{1}{3}(2m^3 - 11m + 12)$.

Problem: Resolve the edge-chromatic number in the unsettled cases. I can use ILP (integer-linear programming) to find colorings and so show that $Q_{3,5}$, $Q_{3,7}$, and $Q_{3,11}$ are class 1. The state of $Q_{3,9}$ is unresolved, so there is a chance it is class 2.

Proofs.

The m, m odd case in WJ's theorem follows from a theorem of Fournier that says that a graph for which the vertices of maximum degree form an independent set is class 1. For $Q_{m,m}$ there is only one vertex of maximum degree, the central vertex.

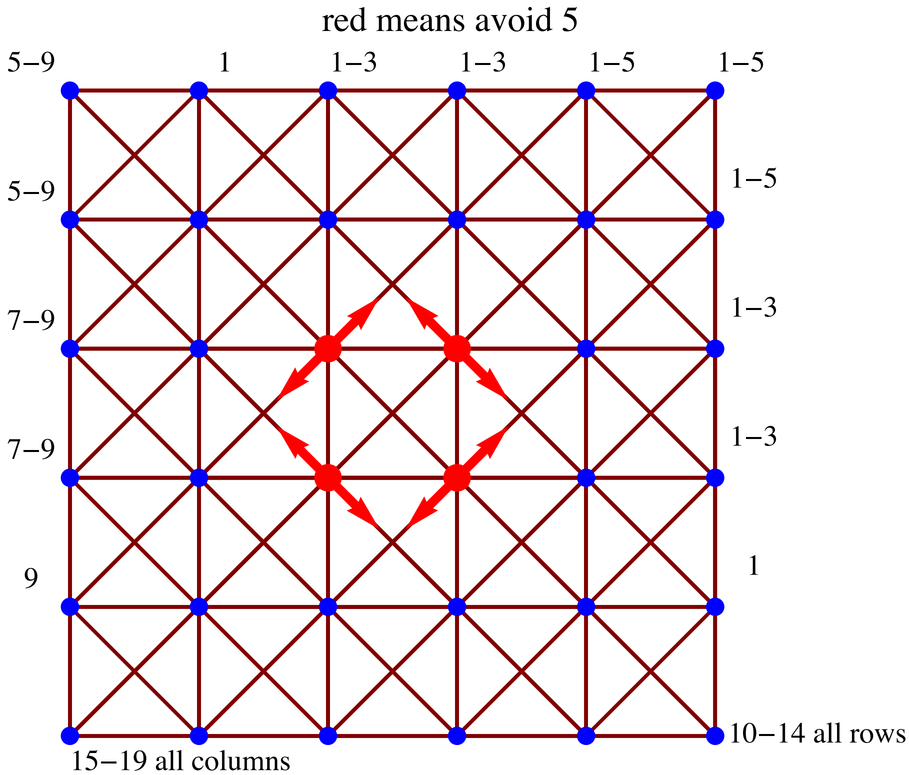
The class two case of WJ's theorem follows from the observation that a matching in $Q_{m,n}$ when m is odd and n even can have (because $m\,n$ is odd) at most $(m\,n - 1)/2$ edges. Combining this with the formulas for the maximum degree and the number of edges means that one cannot cover all edges in d matchings when the stated inequality holds. Thus $Q_{m,n}$ is class 2 for $(m, n) = (3, \geq 13)$, $(5, \geq 71)$, etc. Using ILP (integer-linear programming) to find colorings we showed that $Q_{3,5}$, $Q_{3,7}$, $Q_{3,9}$, and $Q_{3,11}$ are class 1.

For JDV's proof we need the well-known results that: K_n, n odd has edge coloring number n. The max degree is $n - 1$ so this is class 2. Further, the missing colors at each vertex are $1, 2, \ldots, n$ (reason: if any was repeated then some color would never appear as a missing color. That color would then appear at n vertices, and therefore on $\lceil n/2 \rceil$ edges and those edges cannot be a matching. And if n is even then K_n has edge coloring $n - 1$, and so is class 1.

Now the main proofs.

Square case: $n \times n$

Even $n \times n$: Color the up-right diagonals using colors 1 to n. Color the other diagonals using $n - 1$ to $2n - 3$. This works because we can avoid color $n - 1$ at four critical locations, shown by red dots and arrows in the diagram. Then we can use colors $2n - 2$ to $3n - 4$ on the horizontal edges and $3n - 3$ to $4n - 5$ on the verticals.

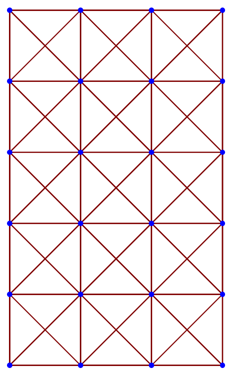


Odd $n \times n$: This follows from the theorem of Fournier that if the set od vertices of maximum degree form a forest, the graph is of type 1. There is but a single vertex of maximum degree in this case. Witold also showed how one can modify Vizing's inductive proof to get a self-contained proof.

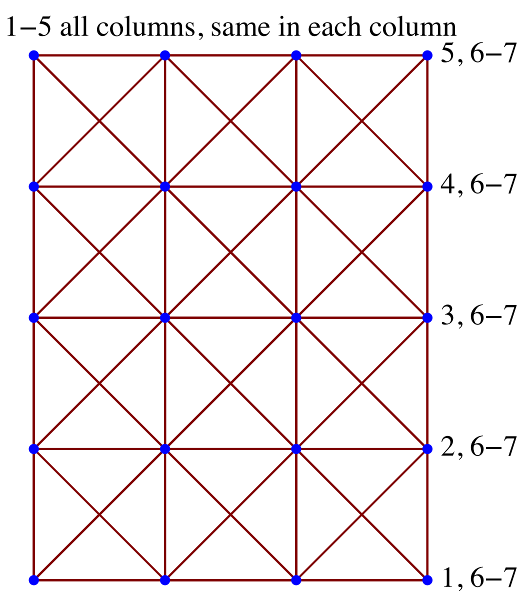
[[Alternatively, the set of vertices of maximum degree consist of just a single vertex. So class one follows from the theorem that if this set is an independent set, or a forest, the graph is class 1.
I have just analyzed the proof of Vizing's theorem (see, e.g., here). It's an induction on the number of edges. The induction step is based on adding one edge to a k+1-colorable graph. The trick is that, in order for the proof to work, if suffices for the maximum degree to be at most k before adding the edge.
So it definitely applies to the odd nxn case - it suffices to remove one of the edges incident to the center square. After the removal, the maximum degree is 4n-5. By "normal" Vizing, we color the graph using 4n-4 colors. Now, we put the edge back, replaying the proof and we get a 4n-4-coloring of the full graph, which proves that it's class 1.]]

Nonsquare: $m \times n$ ($m < n$)

m and n both even: use $m - 1$ colors in each diagonal and on on the horizontals. Use $n - 1$ colors in the other direction. This gives a coloring with $3\,m + n - 4$ colors.



m even, n odd: The longest diagonals have m vertices so are colorable using $m-1$ colors; thus $2m-2$ colors used for the diagonals. In the orthogonal directions, we use only $m+n-2$ colors by this color-sharing strategy: In the vertical (odd) direction, colors 1 to n are used, the same way in each line, so that the missing edge color at any vertex is the same as one moves along any horizontal level. Moreover, the missing colors can be taken to be 1, 2, 3, 4, 5 in order, though this does not really matter. Then for each horizontal level, that missing color is used along with colors $n+1$ to $m+n-2$ to color that horizontal complete graph. Total is $(2m-2)+(m+n-2)$, or $3m + n - 4$



m odd, n even: We use a more involved sharing strategy. The horizontal lines and the longest diagonals each form K_m, and so the edges can be m-colored. So use colors 1 to m for the horizontal lines, $m+1$ to $2m$ for the up-right long diagonals, and $2m+1$ to $3m$ for the up-left long diagonals. In each of these three cases, there is one color not used at each vertex. Choose colors on each of these lines so that the unused colors are constant in the even direction. More precisely, let 1 be the unused color in the first column for the horizontal edges. Let 2 be the unused color in the next column, and so on. The diagram below shows how the missing colors would work in the columns: in the first columns 1, 4,7 are missing at each vertex. In the second column 2, 5, 8 are missing at each vertex, and in the third 3, 6, 9 are missing. But if 3 colors are missing at each vertex, we need only use $n-1-3$ additional colors for the vertical edges in a given column. Thus all the vertical edges can be colored using new colors $3m+1$ to $3m+n-4$, yielding the desired total. Note that this same strategy applies to the m odd, n odd cases, with $n-3$ rather than $n-4$ colors used in each n-length line, and so provides a solution in $3m+n-3$ colors. Vizing's Theorem tells us that such a solution exists, but this method gives a quick construction of the class 2 coloring.

Colors 1, 2, ..., 11