A Heuristic for the Prime Number Theorem

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Why e?

Why does e play such a central role in the distribution of prime numbers? Simply citing the Prime Number Theorem (PNT), which asserts that $\pi(x) \sim x / \ln x$, is not very illuminating. Here "~" means "is asymptotic to" and $\pi(x)$ is the number of primes less than or equal to x. So why do natural logs appear, as opposed to another flavor of logarithm?

The problem with an attempt at a heuristic explanation is that the sieve of Eratosthenes does not behave as one might guess from pure probabilistic considerations. One might think that sieving out the composites under x using primes up to $\sqrt{x}$ would lead to $x \prod_{p<\sqrt{x}} \left(1 - \frac{1}{p}\right)$ as an asymptotic estimate of the count of numbers remaining (the primes up to x; p always represents a prime). But this quantity turns out to be *not* asymptotic to $x / \ln x$. For F. Mertens proved in 1874 that the product is actually asymptotic to $2\, e^{-\gamma} / \ln x$, or about $1.12 / \ln x$. Thus the sieve is 11% (from $1 / 1.12$) more efficient at eliminating composites than one might expect. Commenting on this phenomenon, which one might call the Mertens Paradox, Hardy and Wright [5, p. 372] said: "Considerations of this kind explain why the usual 'probability' arguments lead to the wrong asymptotic value for $\pi(x)$." For more on this theorem of Mertens and related results in prime counting see [3; 5; 6, exer. 8.27; 8].

Yet there ought to be a way to explain, using only elementary methods, why natural logarithms play a central role in the distribution of primes. A good starting place is two old theorems of Chebyshev (1849).

Chebyshev's First Theorem. For any $x \geq 2$, $0.92\, x / \ln x < \pi(x) < 1.7\, x / \ln x$.

Chebyshev's Second Theorem. If $\pi(x) \sim x / \log_c x$, then $c = e$.

A complete proof of Chebyshev's First Theorem (with slightly weaker constants) is not difficult and the reader is encouraged to read the beautiful article by Don Zagier [9] — the very first article ever published in this journal (see also [1, §4.1]). The first theorem tells us that $x/\log_c x$ is a reasonable rough approximation to the growth of $\pi(x)$, but it does not distinguish e from other bases.

The second theorem can be given a complete proof using only elementary calculus [8]. The result is certainly a partial heuristic for the centrality of e since it shows that, if any logarithm works, then the base must be e. Further, one can see the exact place in the proof where e arises ($\int 1/x \, dx = \ln x$). But the hypothesis for the second theorem is a strong one; here we will show, by a relatively simple proof, that the same conclusion follows from a much weaker hypothesis. Of course, the PNT eliminates the need for any hypothesis at all, but its proof requires either an understanding of complex analysis or the willingness to read the sophisticated "elementary proof". The first such was found by Erdos and Selberg; a modern approach appears in [7].

Our presentation here was inspired by a discussion in Courant and Robbins [2]. We show how their heuristic approach can be transformed into a proof of a strong result. So even though our original goal was just to motivate the PNT, we end up with a proved theorem that has a simple statement and quite a simple proof.

Theorem. If $x/\pi(x)$ is asymptotic to an increasing function, then $\pi(x) \sim x/\ln x$.

Figure 1 shows that $x/\pi(x)$ is assuredly not increasing. Yet it does appear to be asymptotic to the piecewise linear function that is the upper part of the convex hull of the graph. Indeed, if we take the convex hull of the full infinite graph, then the piecewise linear function $L(x)$ corresponding to the part of the hull above the graph is increasing (see last section). If one could prove that $x/\pi(x) \sim L(x)$ then, by the theorem, the PNT would follow. In fact, using PNT it is not too hard to prove that $L(x)$ is indeed asymptotic to $x/\pi(x)$ (such proof is given at the end of this paper). In any case, the hypothesis of the theorem is certainly believable, if not so easy to prove, and so the theorem serves as a heuristic explanation of the PNT.

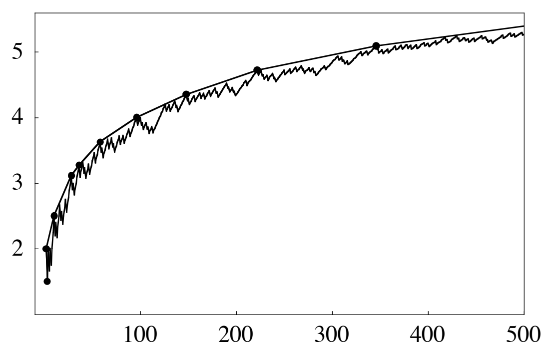

Figure 1. A graph of $x/\pi(x)$ shows that the function is not purely increasing. The upper convex hull of the graph is an increasing piecewise linear function that is a good approximation to $x/\pi(x)$.

Nowadays we can look quite far into the prime realm. Zagier's article of 26 years ago was called *The First Fifty Million Prime Numbers*. Now we can look at the first 700 quintillion prime numbers. Not one at a time, perhaps, but the exact value of $\pi(10^i)$ is now known for i up to 22; the most recent value is due to Gourdon and Sabeh [4] and is $\pi(4 \cdot 10^{22}) = 783\,964\,159\,847\,056\,303\,858$. Figure 2 is a log-log plot that shows the error when these stratospheric π values are compared to $x/\ln x$ and also the much better logarithmic integral estimate li(x) (which is $\int_0^x 1/\ln t \, dt$).

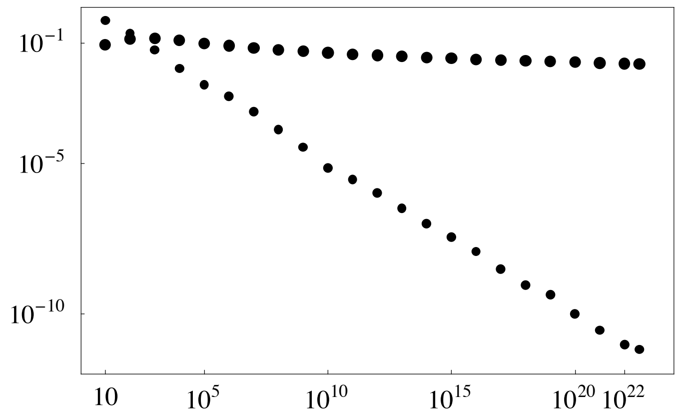

Figure 2. The large dots are the absolute value of the error when $x/\ln x$ is used to approximate $\pi(x)$, for $x = 10^i$. The smaller dots use $\mathrm{li}(x)$ as the approximant.

Two Lemmas

The proof requires two lemmas. The first is a consequence of Chebyshev's First Theorem, but can be given a short and elementary proof; it states that almost all numbers are composite.

Lemma 1. $\lim_{x\to\infty} \pi(x)/x = 0$.

Proof. First use an idea of Chebyshev to get $\pi(2n) - \pi(n) < 2n/\ln n$ for integers n. Take log-base-n of both sides of the following to get the needed inequality.

$$4^n = (1+1)^{2n} > \binom{2n}{n} \geq \prod_{n<p\leq 2n} p > \prod_{n<p\leq 2n} n = n^{\pi(2n)-\pi(n)}$$

This means that $\pi(2n) - \pi(n) \leq (\ln 4)\, n/\ln n$. Suppose n is a power of 2, say 2^k; then summing over $2 \leq k \leq K$, where K is chosen so that $2^K \leq n < 2^{K+1}$, gives

$$\pi(n) \leq 2 + \sum_{k=2}^{K} \frac{2^k \ln 4}{k}$$

Here each term in the sum is at most $3/4$ of the next term, so the entire sum is at most 4 times the last term. That is, $\pi(n) \leq c\, n/\ln n$, which implies $\pi(n)/n \to 0$. For any $x > 0$, we take n to be the first power of 2 past x, and then $\pi(x)/x \leq 2\,\pi(n)/n$, concluding the proof.

By keeping careful track of the constants, the preceding proof can be used to show that $\pi(x) \leq 8.2\, x/\ln x$, yielding one half of Chebyshev's first theorem, albeit with a weaker constant.

The second lemma is a type of Tauberian result, and the proof goes just slightly beyond elementary calculus. This lemma is where natural logs come up, well, naturally. For consider the hypothesis with $\log_c$ in place of ln. Then the constant $\ln c$ will cancel, so the conclusion will be unchanged!

Lemma 2. If $W(x)$ is decreasing and $\int_2^x W(t) \ln(t)/t\, dt \sim \ln x$, then $W(x) \sim 1/\ln x$.

Proof. Let ϵ be small and positive; let $f(t) = \ln(t)/t$. The hypothesis implies $\int_x^{x^{1+\epsilon}} W(t)\, f(t)\, dt \sim \epsilon \ln x$ (to see this split the integral into two: from 2 to x and x to $x^{1+\epsilon}$). Since $W(x)$ is decreasing,

$$\epsilon \ln x \sim \int_x^{x^{1+\epsilon}} W(t)\, f(t)\, dt \le W(x) \int_x^{x^{1+\epsilon}} \frac{\ln t}{t}\, dt = \epsilon\left(1+\frac{\epsilon}{2}\right) W(x)\, (\ln x)^2$$

Thus $\liminf_{x\to\infty} W(x)\ln x \ge 1\Big/\left(1+\frac{\epsilon}{2}\right)$. A similar argument starting with $\int_{x^{1-\epsilon}}^x W(t)\, f(t)\, dt \sim \epsilon \ln x$ shows that $\limsup_{x\to\infty} W(x)\ln x \le 1\Big/\left(1-\frac{\epsilon}{2}\right)$. Since ϵ can be arbitrarily small, we have $W(x)\ln x \sim 1$.

Proof of the Theorem

Theorem. If $x/\pi(x)$ is asymptotic to an increasing function, then $\pi(x) \sim x/\ln x$.

Proof. Let $L(x)$ be the hypothesized increasing function and let $W(x) = 1/L(x)$, a decreasing function. It suffices to show that the hypothesis of Lemma 2 holds, for then $L(x) \sim \ln x$. Let $f(t) = \ln(t)/t$. Note that if $\ln x \sim g(x) + h(x)$ where $h(x)/\ln x \to 0$, then $\ln x \sim g(x)$; we will use this several times in the following sequence, which reaches the desired conclusion by a chain of 11 relations. The notation $p^k \,\|\, n$ in the third expression means that k is the largest power of p that divides n; the equality that follows the $\|$ sum comes from considering each p^m for $1 \le m \le k$.

$$\ln x \sim \frac{1}{x} \sum_{n\le x} \ln n \qquad \text{(can be done by machine; note 1)}$$

$$= \frac{1}{x} \sum_{n\le x} \sum_{p^k \| n} k \ln p = \frac{1}{x} \sum_{n\le x} \sum_{p^m | n,\, m\ge 1} \ln p$$

$$= \frac{1}{x} \sum_{p^m \le x,\, 1\le m} \ln p \left\lfloor \frac{x}{p^m} \right\rfloor \sim \frac{1}{x} \sum_{p^m \le x,\, 1\le m} \ln p \, \frac{x}{p^m} \qquad \text{(error is small; note 2)}$$

$$\sim \sum_{p\le x} f(p) + \sum_{p^m \le x,\, 2\le m} \frac{\ln p}{p^m} \sim \sum_{p\le x} f(p) \qquad \text{(geometric series estimation; note 3)}$$

$$= \pi(\lfloor x \rfloor)\, f(x) - \int_2^x \pi(t)\, f'(t)\, dt \qquad \text{(partial summation; note 4)}$$

$$\sim -\int_2^x \pi(t)\, f'(t)\, dt \qquad \text{(because } \pi(\lfloor x \rfloor)\, f(x)/\ln x \le \pi(x)/x \to 0 \text{ by Lemma 1)}$$

$$\sim -\int_2^x t\, W(t) \left(\frac{1}{t^2} - \frac{\ln t}{t^2}\right) dt \qquad \text{(because } \pi(t) \sim t\, W(t))$$

$$\sim \int_2^x W(t)\, \frac{\ln t}{t}\, dt \qquad \text{(l'Hôpital, note 5)}$$

Notes:

1. It is easy to verify this relation using standard integral test ideas: start with the fact that the sum lies between $\int_1^x \ln t\, dt$ and $x \ln x$. But it is intriguing to see that *Mathematica* can resolve this using symbolic algebra. The sum is just $\ln(\lfloor x \rfloor !)$ and *Mathematica* quickly returns 1 when asked for the limit of $x \ln x/\ln(x!)$ as $x \to \infty$.

2. $$\text{error}/\ln x \le \frac{1}{x \ln x} \sum_{p^m \le x} \ln p \le \frac{1}{x \ln x} \sum_{p\le x} \log_p x \, \ln p = \frac{1}{x \ln x} \sum_{p\le x} \ln x = \pi(x)/x \to 0 \text{ by Lemma 1}$$

3. The second sum divided by $\ln x$ approaches 0 because:

$$\sum_{p^m \le x,\, 2 \le m} \frac{\ln p}{p^m} \le \sum_{p \le x} \ln p \sum_{m=2}^{\infty} \frac{1}{p^m} = \sum_{p \le x} \frac{\ln p}{p(p-1)} \le \sum_{n=2}^{\infty} \frac{\ln n}{n(n-1)}$$
$$\le \sum_{n=2}^{\infty} \frac{(n-1)^{1/2}}{(n-1)^2} < \infty$$

4. We use *partial summation*, a technique common in analytic number theory. Write the integral from 2 to x as a sum of integrals over $[n, n+1]$ together with one from $\lfloor x \rfloor$ to x and use the fact that $\pi(t)$ is constant on such intervals and jumps by 1 exactly at the primes. More precisely:

$$\int_2^x \pi(t) f'(t)\, dt = \int_{\lfloor x \rfloor}^x \pi(t) f'(t)\, dt + \sum_{n=2}^{\lfloor x \rfloor - 1} \int_n^{n+1} \pi(t) f'(t)\, dt$$

$$= \pi(\lfloor x \rfloor)(f(x) - f(\lfloor x \rfloor)) + \sum_{n=2}^{\lfloor x \rfloor - 1} \pi(n)(f(n+1) - f(n)) = \pi(\lfloor x \rfloor) f(x) - \sum_{p \le x} f(p)$$

5. L'Hôpital's rule on $\frac{1}{\ln x} \int_2^x t\, W(t) \frac{1}{t^2}\, dt$ yields $\frac{W(x)/x}{1/x} = W(x) = 1/L(x)$, which approaches 0 by Lemma 1.

No line in the proof uses anything beyond elementary calculus except the call to Lemma 2. The result shows that if there is any nice function that characterizes the growth of $\pi(x)$ then that function must be asymptotic to $x/\ln x$. Of course, the PNT shows that this function does indeed do the job.

This proof works with no change if base-c logarithms are used throughout. But as noted, Lemma 2 will force the natural log to appear! The reason for this lies in the indefinite integration that takes places in the lemma's proof.

Conclusion

Might there be a chance of proving in a simple way that $x/\pi(x)$ is asymptotic to an increasing function, thus getting another proof of PNT? This is probably wishful thinking. However, there is a natural candidate for the increasing function. Let $L(x)$ be the upper convex hull of the full graph of $x/\pi(x)$ (precise definition to follow). The piecewise linear function $L(x)$ is increasing because $x/\pi(x) \to \infty$ as $x \to \infty$. Moreover, using PNT, we can give a proof that $L(x)$ is indeed asymptotic to $x/\pi(x)$. But the point of our work in this paper is that for someone who wishes to understand why the growth of primes is governed by natural logarithms, a reasonable approach is to convince oneself via computation that the convex hull just mentioned satisfies the hypothesis of our theorem, and then use the relatively simple proof to show that this hypothesis rigorously implies the prime number theorem.

We conclude with the convex hull definition and proof. Let B be the graph of $x/\pi(x)$: the set of all points $(x, x/\pi(x))$ where $x \ge 2$. Let C be the convex hull of B: the intersection of all convex sets containing B. The line segment from $(2, 2)$ to any $(x, x/\pi(x))$ lies in C. As $x \to \infty$, the slope of this line segment tends to 0 (because $\pi(x) \to \infty$). Hence for any positive a and ϵ, the vertical line $x = a$ contains points in C of the form $(a, 2 + \epsilon)$. Thus the intersection of the line $x = a$ with C is a set of points (a, y) where $2 < y \le L(x)$. The function $L(x)$ is piecewise linear and $x/\pi(x) \le L(x)$ for all x. This function is what we call the *upper convex hull* of $x/\pi(x)$.

Theorem. The upper convex hull of $x/\pi(x)$ is asymptotic to $\pi(x)$.

Proof. For given positive ϵ and x_0, define a convex set $A(x_0)$ whose boundary consists of the positive x-axis, the line segment from $(0, 0)$ to $(0, (1+\epsilon)(\ln x_0))$, the line segment from that point to $(e\, x_0, (1+\epsilon)(1+\ln x_0))$, and finally the curve $(x, (1+\epsilon)\ln x)$ for $e\, x_0 \leq x$. The slopes match at $e\, x_0$, so this is indeed convex. The PNT implies that for any $\epsilon > 0$ there is an x_1 such that, beyond x_1, $(1-\epsilon)\ln x < x/\pi(x) < (1+\epsilon)\ln x$. Now choose $x_0 > x_1$ so that $\pi(x_1) < (1+\epsilon)\ln x_0$. It follows that $A(x_0)$ contains B: beyond x_0 this is because $x_0 > x_1$; below x_1 the straight part is high enough at $x = 0$ and only increases; and between x_1 and x_0 this is because the curved part is convex down and so the straight part is above where the curved part would be, and that dominates $\pi(x)$ by the choice of x_1. This means that the convex hull of the graph of $x/\pi(x)$ is contained in $A(x_0)$, because $A(x_0)$ is convex. That is, $L(x) \leq (1+\epsilon)\ln x$ for $x \geq x_0$. Indeed, $(1-\epsilon)\ln x \leq x/\pi(x) \leq L(x) \leq (1+\epsilon)\ln x$ for all sufficiently large x. Hence $x/\pi(x) \sim L(x)$.

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