Alphanumeric Divisibility

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Take a word using at most 10 letters, such as *COLLEGEMATHJ*. Can one subsititute distinct digits for the 10 letters so as to get the resulting base-10 number divisible by d? Of course, it depends on d. If d has 100 digits the answer is clearly NO. If $d = 100$ the answer is again NO, since *HJ* cannot be 00. If d is 2, the answer is obviously YES: just use an even digit for *J*; similarly if $d = 5$ or 10. Between these two extremes there are a variety of challenges, depending on d.

A nice special case of this problem, first raised by N. Kildonan [K], is to show that one can always force divisibility by 3. Harder variations are to show that 9 can be forced, or that 7 cannot. Still harder (we needed a computer search to solve it) is to show that 18 can be forced. Perhaps one can prove this by hand. The reader looking for an immediate challenge might try to make *COLLEGEMATHJ* divisible by 18.

To phrase things precisely, let a *word* be any string made from 10 or fewer distinct letters; for each word and each possible substitution of distinct digits for the letters, there is an associated *value*: the base-10 number one gets by carrying out the substitution. If all substitutions yield a value for the word w that is not divisible by d, then w is called a *blocker* for d. If any word ending (on the right) with the word w fails to be divisible by d, then w is called a *strong blocker* for d. An integer d is called *attainable* if every sufficiently long pattern can be made to be divisible by d by some substitution of distinct digits for letters. Thus d is not attainable if there exist arbitrarily long blockers. The use of arbitrarily long strings is important because, for example, *AB* is a blocker for 101, but for a trivial reason.

In this paper we will find all attainable integers; moreover, they are all strongly attainable. Note that any divisor of an attainable number is attainable.

As noted, some cases are extremely easy: 2, 5, and 10 are obviously attainable, and it is just about as easy to get 4 or 8. It takes a little work to get 3 and 9 (proofs given below). Our main theorem resolves the attainability status of all integers.

Theorem 1. The attainable integers consist of all the divisors of 18, 24, 45, 50, 60, or 80.

Before proving this, we discuss the cases of 3 and 9 for completeness and to introduce some ideas common in the later proofs. These results follow from the positive result for 18, but that required a computer search, while 3 and 9 can be done by hand. Of course, we use the well known fact that when d is 3 or 9, then d divides a number iff d divides the sum of its digits.

3 is attainable (N. Kildonan [K]): Given a word, let the 10 letters be grouped as A_i, B_i, and C_i, where each A_i has a multiplicity (perhaps 0) that is divisible by 3, each B_i has a multiplicity of the form $3k+1$, and each C_i has a multiplicity of the form $3k+2$. Let a, b, and c be the number of A_i, B_i, and C_i, respectively. Look for one, two, or three pairs among the B_i and replace them with digits 1 and 2, and 4 and 5 for the second pair, and 7 and 8 for the third pair. Then look for pairs of the C_i and replace them with digits in any of the still-available pairs from (1, 2), (4, 5), (7, 8). These substitutions take care of $B_i \bigcup C_i$ except possibly four letters (since we used three pairs) and we can use 0, 3, 6, 9 for them. The letters A_i can be assigned the remaining digits in any order. This forces the final number to have a digit sum that is divisible by 3.

9 is attainable (R. Israel and R. I. Hess [K]): Suppose a pattern has length n. Suppose some letter occurs k times, where $n-k$ is not divisible by 3. Assign 9 to this letter and 0 to 8 arbitrarily to the other letters. This produces some value $v \pmod 9$. Now replace each digit between 0 and 8 by the next higher digit, wrapping back to 0 in the case of 8. This adds $n-k$ to the mod-9 value. But $n-k$ is relatively prime to 9, so we can do this $-v/(n-k)$ times, where the division uses the mod-9 inverse, in order to get the value 0.

The other case is that every letter has a multiplicity k where $k \equiv n \pmod 3$. If in fact every multiplicity is congruent to $n \bmod 9$, then any assignment will yield a value congruent to $n(0+1+\cdots+9) = 45n \equiv 0 \pmod 9$. If some multiplicity $k \equiv n \pmod 3$ but $k \not\equiv n \pmod 9$ then proceed as in the first half of the proof: assign 9 to this letter, 0 to 8 to the other letters, and then cyclically permute the 0-to-8 values. Each permutation adds $n-k$ and this will eventually transform the value v, which is divisible by 3, to a value divisible by 9, because 3 divides $n-k$, but 9 does not.

Now onto the proof of Theorem 1, which follows from the following lemmas.

Lemma 1. Any integer divisible by a prime greater than 5 is not attainable.
Lemma 2. The largest powers of 2, 3, and 5 that are attainable are 16, 9, and 25, respectively.
Lemma 3. The numbers 36, 48, 75, 90, 100, 120 are not attainable.
Lemma 4. The numbers 18, 24, 45, 50, 60, 80 are attainable.

The ordering of these lemmas indicates how Theorem 1 was found. First the cases of 7 and 11 were settled and that led to the general result of Lemma 1. It followed that the only candidates for attainability had the form $2^a\, 3^b\, 5^c$. Once the powers of 2, 3, and 5 were resolved by Lemma 2, the candidate list reduced to the 45 divisors of $3600 = 16 \cdot 9 \cdot 25$. Resolving the situation for those divisors, with some computer help, led to Lemmas 3 and

4. The theorem follows from the lemmas because Lemmas 3 and 4 settle the status of all the divisors of 3600.

A key idea is that the sum of the ten digits is 45. With that fact, we can start with Lemma 3, which will show how the lack of attainability is proved. Of course, we use here the fact that the mod-9 value of an integer is the mod-9 value of the sum of its base-10 digits, and the same for mod-3 values. We use the letters *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*, *J*, *K* as the ten letters; w^g denotes the concatenation of *g* copies of the string *w*. The following table lists the blockers we need for Lemmas 2 and 3.

d	blocker
27	AAB
32	$ABBAB$
36	$(ABCDEFGHJ)^5\, J^4\, K$
48	$(ABCDEFGH)^2\, KKJK$
75	$AABA$
90	$A^6\,(BCDEFGHJ)^7\, JK$
100	AB
120	$ABCDEFGHJJJK$
125	BBA

Table 1. Blockers to show nonattainability; sometimes two letters are enough, sometimes not.

We show that the words in Table 1 are blockers. The easiest case is 100, since any word ending in *AB* has a value that is not divisible by 100.

36: $(ABCDEFGHJ)^5\, J^4\, K \equiv K + 4\,J + (5\,(45 - K)) \equiv 4\,J - 4\,K \pmod 9$. The only way $4\,(J - K)$ is divisible by 9 is if *JK* is either 90 or 09, and neither is divisible by 4. Extension on the left by $A^{9\,i}$ preserves the mod-36 value because 111111111 is divisible by 9.

48: The rightmost 4 digits of $(ABCDEFGH)^2\, KKJK$ must be one of 0080, 2272, 4464, 6656, 8848, as these are the only *KKJK* patterns that are divisible by 16. But the mod-3 value of the string is then one of the following, where we work with vectors and ignore *K* which occurs three times:

$$2\,(45 - (\{8, 7, 6, 5, 4\} + \{0, 2, 4, 6, 8\})) + \{8, 7, 6, 5, 4\} = \{82, 79, 76, 73, 70\},$$

and none is divisible by 3. Left extension by $A^{3\,i}$ preserves the mod-48 value.

75: $AABA \equiv 51\,A + 10\,B \pmod{75}$. Multiplying by 53 transforms the condition to $3\,A + 5\,B \equiv 0 \pmod{75}$. $3 \le 3\,A + 5\,B \le 69$, so the condition is never true. Left extension by $A^{3\,i}$ preserves the mod-75 value.

90: $A^6(BCDEFGHJ)^7\, JK \equiv 6\,A + 7\,(45 - A) + J \pmod 9$ because *K* must be 0. The expression simplifies to $J - A$ which cannot be divisible by 9 because 0 is already assigned to K. Left extension by $A^{9\,i}$ preserves the mod-90 value.

120: $ABCDEFGHJJJK \equiv 45 + 2\,J \equiv 2\,J \pmod 3$. But JJK must be either 440 or 880 to get divisibility by 40; therefore $2\,J$ is either 8 or 16, and so is not divisible by 3. Left extension by $A^{3\,i}$ preserves the mod-120 value.

32: Any word ending in $ABBAB$ has a value that, reduced mod 32, is $10\,010\,A + 1101\,B \equiv 26\,A + 13\,B \pmod{32}$. If this is congruent to 0 (mod 32), then 13 cancels, leaving $2\,A + B$. But this sum is between 1 and 18 + 8 = 26, and so is not divisible by 32.

125: A number is divisible by 125 iff it ends in one of 125, 250, 375, 500, 625, 750, 875, or 000. Thus BBA is a strong blocker for 125.

27: The value of AAB is $110\,A + B \equiv 2\,A + B \pmod{27}$. But $1 \le 2\,A + B \le 26$ so this is not divisible by 27. And this is a strong blocker because we can prepend $A^{27\,i}$ which leaves the mod-27 value of the word unchanged.

Next we prove Lemma 1. Our first proof of this was a little complicated, but when we focused on words involving two letters only we discovered Theorem 2, which yields Lemma 1 in all cases except 7. Recall Euler's theorem that $a^{\phi(d)} \equiv 1 \pmod d$ when $\gcd(a,\, d) = 1$. It follows that if d is coprime to 10, then there is a smallest positive integer, denoted $\text{ord}_d(10)$, such that $10^{\text{ord}_d(10)} \equiv 1 \pmod d$.

Theorem 2. Let d be coprime to 10 and greater than 10; let $e = \text{ord}_d(10)$. Then $w = A^{e-1}\,B$ is a blocker for d.

Proof. If 3 does not divide d then the value of w is

$$B + \sum_{i=1}^{e-1} 10^i = B - A\ + A\,\frac{10^e - 1}{9} = B - A\ + A\,\frac{K\,d}{9} \equiv B - A\ + A\,d\,\frac{K}{9} \equiv B - A \pmod d;$$

because $d > 10$, d cannot divide $B - A$. Suppose 3 divides d and $d > 81$. Suppose the value of w, in the formula just given, is a multiple of d. Then multiplying by 9 yields $9\,(B - A) + A\,(10^e - 1) = 9\,K\,d$. We learn from this that d divides $9\,(B - A)$. But then $A \ne B$ and $-81 \le 9\,(B - A) \le 81$, contradicting $d > 81$.

There remain the cases that $11 \le d \le 81$ and 3 divides d: 21, 27, 33, 39, 51, 57, 63, 69, 81. In all cases except 21, 27 and 81, $\text{ord}_d\,10 = \text{ord}_{3\,d}\,10$. This means that from $B - A + A\,(10^e - 1)/9 = K\,d$ we can conclude $9\,(B - A) + A\,(10^e - 1) = 3\,K\,(3\,d)$, whence $3\,d$ divides $9\,(B - A)$. So d divides $3\,(B - A)$ whic means that $d \le 27$, contradiction. For the remaining cases:

21: value of w is $B - A$, which cannot be divisible by 21;
27: value of w is $2A + B$, and $1 \le 2A + B \le 26$;
81: value of w is $8A + B$, and $1 \le 8A + B \le 80$.

The preceding result blocks all primes greater than 10. For Lemma 1 we need to deal also with $d = 7$. One can give an alternate construction in the general case that includes 7, and we give the following without proof.

Proposition. Suppose d is coprime to 30 and $e = \mathrm{ord}_d(10)$; let w be the word

$$K\ J K^e\ \ H K^e\ \ G K^e\ \ F K^e\ \ E K^e\ \ D K^e\ \ C K^e\ \ B K^e\ \ A\ K^e.$$

Then the value of w after any substitution is congruent to $9\,(d+1)/2 \pmod d$ and is therefore not divisible by d.

In the case of 7 the long word of the proposition has length 64. Here is a much shorter blocker. The value of *OLD IDAHO USUAL HERE* is always $3 \cdot 45 \pmod 7$ and so this 17-character word is a blocker for 7. And it can be made arbitrarily long by prepending E^6.

On to Lemma 2. The positive results for 16 and 25 are not too difficult (for 25 just use 00 or 25 for the rightmost digits; for 16, enumerate all the 15 possible patterns of length four and check that they all occur in numbers below 10000; see the argument for 80 in next paragraph); the case of 9 was discussed earlier, as were the negative results for 32, 27, and 125.

So only Lemma 4 remains. The proof of that was carried out using *Mathematica*, which could check the many cases (several hundred thousand) in a few minutes. Some values are easy by hand, such as 50 and 80: For 50 just use 00 or 50 for the rightmost two digits. For 80, observe that the following list of numbers divisible by 80 contains all 15 patterns of length 4, and divisibility by 80 is not affected by digits farther left:
0000, 8880, 0080, 8000, 0800, 8800, 8080, 0880, 1120, 2320, 1440, 0160, 1040, 1200, 1280. For the remaining cases, 18, 24, 45, and 60, the following algorithm was implemented. We present the main ideas using $d = 18$.

Suppose $d = 18$ and consider the sample word $w = K^7\,A\,A\,B\,B\,C\,C\,D^{95}\,E^{95}\,F^{45}\,G^{45}\,H^5\,J^5\,K$. Ignore K for a moment, since its assignment is restricted to even digits. And keep in mind that we want this number to be divisible by 9, which is determined by the sum of the digits — positions are irrelevant. The multiplicities of the other letters are (2, 2, 2, 95, 95, 45, 45, 5, 5). So the multiplicity vector, which places each multiplicity into a mod-9 bin, is (2, 0, 3, 0, 0, 4, 0, 0, 0), where the 2 comes from the two 45s, which are 0 mod 9. This is a composition of 9 into 9 parts. The mod-9 value of w under the assignment letter$_i \to b_i$, will be $0\,(b_6 + b_7) + 2\,(b_1 + b_2 + b_3) + 5\,(b_4 + b_5 + b_8 + b_9) + 8\,b_{10}$, and we want this to be divisible by 9, under the restriction that b_{10} is even. Note that the search for a successful digit assignment can work within the batches with the same multiplicity, since within such a batch the digit-order does not change the value. So we seek an assignment so that the expression above is divisible by 9. An alternative is to check all permutations, but the batch method is much faster. Because there are 24310 compositions of 9 into 9 parts, taking into account the 9 possible multiplicities for the special digit *K* gives 218 790 multiplicity vectors that must be dealt with. The assignment-search step then required 11 093 273 trials in all; thus on average about 50 assignments were checked for each multiplicity vector.

For a second example, consider $d = 60$. Here the patterns are *JK*, *KK*, and we proceed as with 18, treating each pattern as a separate case, searching for divisibility by 3, and enforcing divisibility by 20 in the assignments to the five patterns (that is, *K* must be 0 and the tens digit must be one of 0, 2, 4, 6, 8. Here then is the general algorithm.

Attainability Algorithm for $d \in \{18, 24, 45, 60\}$

Step 1. Enumerate the patterns at the right end of the word that will enforce divisiblity by the parts of d that are powers of 2 or 5. For 18 or 45, there is only one, *K*; for 60, *JK, KK*; and for 24, *JKK*, *KKK*, *JJK*, *KJK*, *HJK*.

Step 2. Enumerate the digits that can be used on the pattern(s) of Step 1. They are {2, 4, 6, 8, 0} for 18, and various other combinations for the other three cases.

Step 3. If 9 divides d, enumerate the multiplicity sequences $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ for the possible multiplicities of each of the 10 letters; here a_i represents the number of letters whose multiplicity is $i \bmod 9$. Each a_i can be reduced mod-9, so lies in {0, 1, …, 8}. If only 3 divides d, the approach is similar but which vectors of length 3 and working mod-3.

Step 4. For each multiplicity vector of Step 3, search for a permutation p of the 10 digits that incorporates the restriction of Step 2 and gets the quantity $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) \cdot p$ to be divisible by 9 (or $(a_0, a_1, a_2) \cdot p$ to be divisible by 3), where a_i is the number of letters whose multiplicity is $i \bmod 3$.

One might well ask if one can have complete confidence in such a complicated program. But if the potential divisor turns out to be not attainable, this algorithm will discover that and return a blocking word. Several of the blocking words of Table 1 were found by this program. Another check occurs by computing the total number of cases in two ways, one using a count on the number of compositions, and the other by inserting a counter in the program.

This need for a computer is not entirely satisfactory since if one wishes to make a given word divisible by, say, 18, one can easily compute the multiplicity function, but then a computer search is needed to find a solution, using an even digit in the units place. Here is how to make *COLLEGEMATHJ* divisible by 18: Turn it into 295543481760. The search takes only a second using the batch approach described above. In fact, almost 6% of the 10! permutations work, so one can find a good letter assignment by a straightforward search among the permutations. Still, a more constructive method, such as that presented for $d = 9$, would be a nice improvement to our proof of Theorem 1. On the other hand, our theorem does reveal the complete story regarding which numbers are attainable.

A Small Alphabet

To conclude, we will solve an interesting variant, where the alphabet is restricted to the two letters A and B. We use the terms 2-attainability and 2-blocker for this context. Theorem 3 shows that with this smaller family of words there are a few more attainable numbers: 27 in this case compared to 22 in the unrestricted case.

Theorem 3. The set of 2-attainable numbers is the set of all divisors of 24, 50, 60, 70, 80, or 90.

Proof. The positive results are easy: Assume the units digit is B. Let $B = 0$ and $A = 6, 5, 6, 7, 8$, or 9 to get the six attainables of the theorem in order. Theorem 2, combined with the 2-blockers given earlier, shows that we need only consider the 90 divisors of $2^4\,3^2\,5^2\,7$. The theorem also eliminates $d = 21$ or any multiple of it. This filtering leaves only 33 numbers to decide. A search for blockers of the form A^iB succeeds for 36, 56, 100, 120, 140, 150, 175, and 225 (see Table 2, where unneeded ones are omitted). Widening the search to A^iBA yields blockers for 28, 48, and 75. Combining all these cases leaves no d-value undecided.

d	2 -blocker
28	$A^5\,B\,A$
36	AAB
48	$AAAABA$
75	$AABA$
100	AB
120	AAB
175	AB

Table 2. Blockers using only two letters.

The verifications in Table 2 are straightforward:

28: the value of w in the table yields $5\,A + 10\,B \equiv 0 \pmod{28}$, which is the same as $A + 2\,B \equiv 0 \pmod{28}$, impossible because $A + 2\,B \leq 26$.

36: value condition becomes $2\,A + B \equiv 0 \pmod{36}$, and $2\,A + B$ is too small.

48: value condition becomes $A + 2B \equiv 0 \pmod{48}$, and $A + 2B$ is too small.

75: discussed in the unrestricted case.

120: value condition becomes $-10A + B \equiv 0 \pmod{120}$, or. But $-82 \le B - 10A \le 9$ and $B \ne 10A$, so $B - 10A$ cannot be divisible by 120.

175: value condition is $10A + B \equiv 0 \pmod{120}$, and $10A + B \le 98$, so is too small.

Of course we want arbitrarily long blockers, but in all the cases of Table 2 the mod-d value of the word is preserved by prepending an appropriate power of A. In fact, A^{18} works in all cases.

The natural question for further study is to investigate other bases. It might be difficult to investigate very large ones, but there could be some interesting results for smaller bases or prime bases. We leave such work to the interested reader, observing only that base $b = 2$ is not terribly interesting because the only attainable numbers are the obvious ones: 1 and b. The same is true for $b = 3$. Are there prime bases for which the situation is more complicated? We also repeat the question of whether there are simpler algorithms in some cases: Can one show that 18 is attainable without using a computer to check thousands of cases? Still in base 10, one can ask exactly how things get more complicated as the number of letters increases: for example, at which point between two and ten letters does 7 (or 14, 35, 70, or 90) become unattainable?

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Reference

[K] N. Kildonan, Problem 1859, *Crux Mathematicorum*, **20**:6, June 1994, 168-170.