# Some problems and conjectures

Erik Vigren
IRF, Uppsala, Sweden. E-mail: erik.vigren@irfu.se

Through twenty entries I present here some mathematical problems[1] and conjectures. Maybe a teacher could find a couple of the problems suitable to display in courses on e.g., number- or probability theory? Perhaps are there possibilities to expand on some of the results? I would appreciate[2] feedback so that a future updated version of this work may come with e.g., more pedagogical/stringent solutions to some of the problems, remarks addressing earlier works covering similar results and rigorous proofs for or against some of the conjectures.

E. V.
October 2024

**CONTENT**



[1] Some of the problems were earlier rejected when submitted as problem proposals to the American Mathematical Monthly. I share in the beginning of associated entries reasons for the rejections.

[2] I am thankful to Stan Wagon, Roberto Tauraso, Anders Eriksson, Felix Lundberg (particularly with respect to entries 3 and 15) and Andreas Dieckmann (particularly with respect to entries 5-7) for inspiration, numerical checks, advices and/or discussions from time to time over the last couple of years.

**ENTRY 1:** *Leaning sticks problem*

The following problem proposal rightfully struck a reviewer as an example of a solution in search of a problem.

Problem 1.1: For a positive integer $n$, let $b_1, \dots, b_n$ be real numbers from $(0,1)$, let $a_0 = 1$ and set $a_k = a_{k-1}\left(\sqrt{1-a_{k-1}^2 b_k^2} - b_k\sqrt{1-a_{k-1}^2}\right)$ for $1 \le k \le n$. Find the minimum real number, $\varepsilon$, assuring that $\sum_{k=1}^{n} a_k b_k < \varepsilon$ regardless the value of $n$.

*Solution*: The answer is $\varepsilon = \pi/2$. View $a_k$ and $b_k$ respectively as height and base in triangle $T_k$ constructed as follows. Make $n$ dots, $p_1, \dots, p_n$, on the positive $x$-axis $(y = 0)$, the first at $x = b_1$, the second at $x = b_1 + b_2$ and so on. From point $p_1$ draw a line $l_1$ of unit length so that its other end goes to $x = 0$, $y = a_1 = \sqrt{1-b_1^2}$ . Then, for $k = 2, \dots, n$ from $p_k$ draw the line $l_k$ of unit length so that its other end touches the line $l_{k-1}$. The $n$ lines form together with the axes of the coordinate system $n$ triangular regions with individual areas $A_k$. That $a_k$ relate to $a_{k-1}$ and $b_k$ via the recursion formula in the problem statement can be worked out by trivial geometry and algebra (draw a figure!). The triangle $T_k$ consists of one line with length 1, one line with length $b_k$ and one line with length $u_k$. Let $\theta_K$ be the angle opposite to the side of length $b_k$. The value of $\varepsilon$ is two times the minimum area non-reachable by the sum of the triangle areas $A_k$. From the geometric construction can be realized that $\sum_{k=1}^{n} \theta_k < \frac{\pi}{2}$ and since $\sin\theta_k < \theta_k$ for $0 < \theta_k$ follows that

$$\sum_{k=1}^{n} A_k = \frac{1}{2}\sum_{k=1}^{n} a_k b_k = \frac{1}{2}\sum_{k=1}^{n} u_k \sin\theta_k < \frac{1}{2}\sum_{k=1}^{n} u_k\theta_k < \frac{1}{2}\sum_{k=1}^{n} \theta_k < \frac{\pi}{4}$$

Nothing hinders to consider a very large $n$ and "select" $u_k$:s arbitrarily close to 1 and so it is not possible to reduce $\varepsilon$ further.

While working with a modified version of the problem I encountered an obstacle leading to the following conjecture.

Conjecture 1.1: If for any positive real number $c$ smaller than the positive integer $n$ we define

$$f_c(n) = \sqrt{1-\left(1-\frac{c}{n}\right)^{2n}} + \frac{c}{n}\sum_{k=1}^{n-1}\sqrt{1-\left(1-\frac{c}{n}\right)^{2k}},$$

then $f_c(n) < f_c(n+1) < \cosh^{-1} e^c$.

**ENTRY 2:** *An inequality involving the arithmetic-geometric mean*

The arithmetic-geometric mean of the non-negative numbers $x$ and $y$ is denoted $\mathrm{AGM}(x,y)$ and defined as the limiting value of each of the intertwined sequences $a_k = (a_{k-1} + b_{k-1})/2$ and $b_k = \sqrt{a_{k-1}b_{k-1}}$ with $a_0 = x$ and $b_0 = y$. The following conjecture seems hard to prove (or disprove). I gave up trying already after an hour.

Conjecture 2.1: For real numbers $a$, $b$, $c$ with $0 < a < b$ and $\mathrm{AGM}(a,b) = c$ is conjectured that $\mathrm{AGM}\big(\mathrm{AGM}(a,c), \mathrm{AGM}(b,c)\big) > c$.

Remark 2.1: If AGM is replaced by the arithmetic-harmonic mean (which is the same as simple geometric mean) or any regular Pythagorean mean then should $> c$ be replaced by $= c$. These equalities are fairly straightforward to prove. If instead AGM is replaced by the geometric-harmonic mean, then should (again conjectured) $> c$ be replaced by $< c$.

**ENTRY 3:** *A modified version of Langton's ant and the sum of even squares*

Consider an ant (Langton's ant) placed in a square grid with initially only white cells. The ant is initially facing right. The ant notices the color of the cell, turns 90° right/left if the cell color is white/black, recolors the cell to black/white and moves forward by one step. The pattern is repeated. Famously the journey of the ant (and the evolution of the landscape) looks chaotic to begin with while after roundabout 10,000 steps a recurring pattern emerges with the ant (for each period of the pattern) moving further and further away from the starting cell. Many variants of Langton's ant have been looked at but the following result seems not to have attracted attention.

Conjecture 3.1: A flea follows the same rules as Langton's ant but with one exception; the flea jumps forward by 2 units instead of 1 whenever having recolored a cell from black to white. It is conjectured that the flea returns to the starting cell for the $m$:th time following a number of jumps corresponding to the sum of the $m$ first even squares.

Remark 3.1: The conjectured result can be generalized as to concern any flea that jumps $a$ or $b$ units forward depending on the type of recoloring. As long as $a \neq b$ are positive integers the flea returns to the starting cell for the $m$:th time following a number of jumps corresponding to the sum of the $m$ first even squares.

Remark 3.2: The flea follows a peculiar and repetitive pattern. Based on this I have outlined a proof of the conjecture but it is tedious and unlikely to meet the criteria for being classified as stringent.

**ENTRY 4:** *Sum of lengths of sequences within a set of sequences*

Despite repeated failures the conjecture that follows has been entertaining trying to prove.

Definition 4.1: A vector (or finite sequence) belongs to $V_{m,n}$ if it consists of $n$ elements such that each element belongs to $\{1,2,\dots,m\}$.

*Example*: The vector [1 4 2 1 3] can be said to belong to $V_{4,5}$. It also belongs to e.g., $V_{14,5}$.

Definition 4.2: For a vector (or finite sequence), $v$, we let $f(v)$ denote the number of 1:s in the longest sequence of consecutive 1:s when the elements of $v$ are placed clockwise in a circle. A given element equal to 1 can at most be counted once and so $f(v)$ cannot exceed the number of elements within $v$.

*Examples*: With $v_1 = [1\ 4\ 2\ 1\ 3]$, $v_2 = [1\ 1\ 2\ 1\ 3]$ and $v_3 = [1\ 1\ 2\ 1\ 1]$ we get $f(v_1) = 1$, $f(v_2) = 2$ and $f(v_3) = 4$.

Definition 4.3: For positive integers $m$ and $n$ we define $W_{m,n}$ as

$$W_{m,n} = \sum_{v \in V_{m,n}} f(v)$$

Conjecture 4.1: For any prime number $p$ is $W_{m,p}$ a multiple of $mp$.

*Examples:* The only $v$ of $V_{1,3}$ is [1 1 1] and $f(v) = 3$ which is a multiple of $mp$ given that we have $m = 1$ and $p = 3$. The eight $v$:s of $V_{2,3}$ are [1 1 1], [1 1 2], [1 2 1], [1 2 2], [2 1 1], [2 1 2], [2 2 1] and [2 2 2] and the corresponding $f(v)$:s are 3, 2, 2, 1, 2, 1, 1 and 0 which sum to $W_{2,3} = 12$ which is a multiple of $mp = 6$ . In the case of $V_{2,4}$ we get that $W_{2,4} = 30$ (see for your self!). This is not a multiple of 8, but 4 is not a prime number, so it does not conflict with the conjecture.

**ENTRY 5:** *A generalization of a sequence of Leibniz plus a mysterious constant*

The following results were found in the spirit of experimental mathematics. They were in part presented as by-findings in an article (E. Vigren & A. Dieckmann, 2020, Symmetry, 12, 1040) focusing on electric fields in certain line charge configurations. The results are repeated here for further visibility.

Conjecture 5.1: It is known (an unusual way to present Leibniz' sequence for $\pi$) that

$$\frac{1}{\pi} \sum_{k=-\infty}^{\infty} (-1)^k \frac{1/2 - k}{(1/2 - k)^2} = 1 = G^0$$

and it is conjectured that

$$\frac{1}{\pi} \sum_{k,m=-\infty}^{\infty} (-1)^{k+m} \frac{1/2 - k}{(1/2 - k)^2 + m^2} = G = G^1$$

$$\frac{1}{\pi} \sum_{k,m,n,t=-\infty}^{\infty} (-1)^{k+m+n+t} \frac{1/2 - k}{(1/2 - k)^2 + m^2 + n^2 + t^2} = G^2$$

where $G$ is Gauss's constant ($G = 1/\mathrm{AGM}(1, \sqrt{2})$).

Remark 5.1: Despite extensive detective work I have not been able to find a closed-form expression for the "point charge constant"

$$\sum_{k,m,n=-\infty}^{\infty} (-1)^{k+m+n} \frac{1/2 - k}{((1/2 - k)^2 + m^2 + n^2)^{3/2}} \approx 6.19918073 \ldots,$$

but noticed that if replacing $(-1)^{k+m+n}$ by 1 in the triple sum renders the result $2\pi/3$.

**ENTRY 6:** *A multifactorial problem*

This is an example of a constructed problem. Note that the number of exclamation marks in a multifactorial expression tells the separation between numbers to be multiplied. For instance, we have that $33!!!!! = 33\times 28\times 23\times 18\times 13\times 8\times 3 = 119351232$.

Problem 6.1: Evaluate

$$\sum_{k=0}^{\infty}\frac{k^7-k^3-15k^2-75k-125}{(k!!!!!)^4}$$

*Solution*: Note that $k!!!!! = k$ for $1 \le k \le 4$ so that

$$\sum_{k=0}^{\infty}\frac{k^7}{(k!!!!!)^4}=\sum_{k=0}^{\infty}\frac{k^{4+3}}{(k!!!!!)^4}=\sum_{k=1}^{4}k^3+\sum_{k=5}^{\infty}\frac{k^3}{((k-5)!!!!!)^4}=100+\sum_{k=0}^{\infty}\frac{(k+5)^3}{(k!!!!!)^4}$$

$$=100+\sum_{k=0}^{\infty}\frac{\left[\binom{3}{0}5^3k^0+\binom{3}{1}5^2k^1+\binom{3}{2}5^1k^2+\binom{3}{3}5^0k^3\right]}{(k!!!!!)^4}=100+\sum_{k=0}^{\infty}\frac{125+75k+15k^2+k^3}{(k!!!!!)^4}$$

which shows that the sum in the problem statement equals 100.

**ENTRY 7:** *Evaluation of a class of alternating series*

The following (possibly known) result was spotted while contemplating certain lattice sums, but I omit describing the connection here. The idea is presented in some further detail online[3].

Definition 7.1: For nonnegative integers $n$, define $S_n = \sum_{k=0}^{\infty}(-1)^k k^n$.

Remark 7.1: It is known (from Abel regularization) that $S_0 = \frac{1}{2}, S_1 = -\frac{1}{4}, S_2 = 0, S_3 = \frac{1}{8}, \ldots$

Conjecture 7.1: For positive integers $m \ge 1$ holds true that

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}\prod_{s=1}^{m}(k+s) = (m-1)!\,\frac{2^m-1}{2^m}$$

Remark 7.2: Conjecture 7.1 allows to calculate $S_n$ recursively.

*Exercise*: Show that Conjecture 7.1 is consistent with $S_3 = 1/8$.

Remark 7.3: An induction proof of Conjecture 7.1 would be straightforward to complete should it be (maybe it is!?) well established that for $m \ge 1$:

$$\sum_{k=0}^{\infty}(-1)^k\prod_{s=1}^{m}(k+s) = \frac{m!}{2^{m+1}}$$

[3] http://www-elsa.physik.uni-bonn.de/~dieckman/InfProd/InfProd.html

**ENTRY 8:** *Water transfer problem*

A round table problem follows that includes an open part (b). I realize that the solution to part (a) is non-stringent and hence I refer to the solution as sketchy.

Problem 8.1: A liter of water is distributed in some fashion into the initially empty glasses $g_1, g_2, \ldots, g_n$ which are placed in circle around a table (each glass is big enough to contain a liter and there is no spilling in the considered scenario). Transfer $r \geq 1$ then involves moving a fraction of $1/r$ of the content in glass $g_k$ into glass $g_{k+1}$ if $k \neq n$ and into glass $g_1$ if $k = n$ where $k$ is the minimal positive integer that can be expressed as $k = r - bn$ while restricting $b$ to be a nonnegative integer.

(a) Show that the water content in each and every glass approaches $1/n$ liter as $r \to \infty$.

(b) Given $n \geq 3$, is it possible that each glass contains exactly $1/n$ liter after a positive number of moves have been made?

*Sketchy solution*: (a) Let $v_k$ and $v_{k+1}$ be the volume of water in glasses $g_k$ and $g_{k+1}$, respectively, prior to transfer from $g_k$ to $g_{k+1}$ in move $t$ and from $g_{k+1}$ to the next glass in move $t + 1$. Let $v'_{k+1}$ be the volume in $g_{k+1}$ after move $t + 1$. We have that

$$v'_{k+1} = \frac{v_k + tv_{k+1}}{t+1}$$

so if $v_k \geq v_{k+1}$ then is

$$v'_{k+1} \geq \frac{v_{k+1} + tv_{k+1}}{t+1} = v_{k+1}$$

and if $v_k < v_{k+1}$ then is

$$v'_{k+1} < \frac{v_{k+1} + tv_{k+1}}{t+1} = v_{k+1}$$

This implies an evening out tendency for the water content in the glasses, which can be viewed as powered by a flow of water circulating across the glasses. Owing to the divergence of the series $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$ it can be realized that the flow in fact is eternal. From physical intuition is then clear that any given water molecule (or entity) in a far distant future (extremely many moves ahead) is equally probable to be in any given glass. Part (b) is open, not sure about the answer, and no solution is offered here.

Conjecture 8.1 (possibly quite easy to prove): With only two glasses and with transfer of $1/r^2$ instead of $1/r$, the fraction of the water content in $g_2$ to the total water content in $g_1 + g_2$ approaches ln 2.

Conjecture 8.2: With only four glasses, starting with $1/4$ liter in each of $g_1$, $g_2$, $g_3$ and $g_4$ and going for the $1/r^2$ instead of $1/r$ type of transfer, the fraction of the water content in $g_2$ to the total water content in all four glasses approaches $2 - \pi^2/6$.

**ENTRY 9:** *A duality problem*

The problem that follows was deemed too simple but the observation was credited as nice.

Definition 9.1: For $0 \leq k \leq n$, we define

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = n!/\big((n-k)!\,k!\big)$$

and

$$\boxed{\binom{n}{k}} = T_n - (T_{n-k} + T_k)$$

where $T_j = 0 + 1 + 2 + \cdots + j$.

Remark 9.1: Properties such as symmetry and monotonicity hold for $\boxed{\binom{n}{k}}$ and one may show by induction (left out here) that e.g.,

$$\sum_{k=0}^{n} \boxed{\binom{n}{k}} = \frac{n(n-1)(n+1)}{6}$$

$$\sum_{k=0}^{n} \boxed{\binom{n}{k}}^2 = \frac{n^5 - n}{30}$$

Problem 9.1: Consider the following well-known identities:

(i) $\binom{n}{k} = (n/(n-k))\binom{n-1}{k}$ (ii) $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$

where $0 \leq k \leq m \leq n$. Would the relations hold also if all division and multiplication signs, also those not explicitly shown, were to be replaced by minus and plus signs, respectively?

*Solution*: The answer is yes. The modified form of $\binom{n}{k}$, is exactly $\boxed{\binom{n}{k}}$ as described above and

$$\boxed{\binom{n}{k}} = T_n - (T_{n-k} + T_k) = \cdots = (n-k)k$$

where we have left out some algebraic steps. (i) The relation

$$\binom{n}{k} = (n/(n-k))\binom{n-1}{k}$$

becomes in "modified" form

$$\boxed{\binom{n}{k}} = (n - (n-k)) + \boxed{\binom{n-1}{k}} = k + \boxed{\binom{n-1}{k}}$$

This holds because

$$\boxed{\binom{n}{k}} = (n-k)k = \big((n-1)-k\big)k + k = \boxed{\binom{n-1}{k}} + k$$

Part (ii) is left as an exercise.

**ENTRY 10:** *A mathematical spacing problem*

A short article version of this problem was rejected (many years ago) as the topic of mathematical spacing had been extensively studied for a long time. If a reader is aware of previous occurrences of this problem (or equivalent versions thereof) I would like to know so that I can add references as a remark.

Problem 10.1: Let $n \geq 2$ be an integer and let $b < 1$ be a positive real number. Select $n$ real numbers randomly from $(0,1)$ and then label them so that $x_1 \leq x_2 \leq \cdots \leq x_n$. Let $P(n,b)$ be the probability of the event that $x_{j+1} - x_j \leq b$ for all $j \in \{1,2,\dots,n-1\}$. Show that

$$P(n,b) = \sum_{k=0}^{\min\{\lfloor b^{-1}\rfloor, n-1\}} (-1)^k \binom{n-1}{k} (1-kb)^n$$

*Solution*: Instead of points distributed on the line from 0 to 1 we shall view the problem as concerning points distributed on a line of length $L$ and where the separation between neighboring points is satisfactory if it does not exceed $b/L$. The sought probability is the ratio of a volume, $V_D(n,b,L)$, of satisfactory configurations and the volume of the $n$ −dimenisonal cube of side length $L$. As the latter volume is $L^n$ our task is then to show that

$$V_D(n,b,L) = L^n \sum_{k=0}^{\min\left\{\left\lfloor \frac{L}{b}\right\rfloor, n-1\right\}} (-1)^k \binom{n-1}{k}\left(1-\frac{kb}{L}\right)^n = \sum_{k=0}^{\min\left\{\left\lfloor \frac{L}{b}\right\rfloor, n-1\right\}} \mu_k (L-kb)^n$$

where we define $\mu_k = (-1)^k\binom{n-1}{k}$. We shall prove the formula through induction and make use of $\nu_k = (-1)^k\binom{n-2}{k}$ along with the identities (prove these as an exercise!): (i) $\nu_0 = \mu_0$, (ii) $-\nu_{n-2} = \mu_{n-1}$, (iii) $\nu_k - \nu_{k-1} = \mu_k$ for $1 \leq k \leq n-2$.

We note from calculus that

$$V_D(2,b,L) = 2\left(\int_0^{L-b} b\,dx + \int_{L-b}^{L} (L-x)\,dx\right) = 2bL - b^2$$

which can be seen to agree with the formula (note that in this case with $n = 2$ holds for sure true that $\min\left\{\left\lfloor \frac{L}{b}\right\rfloor, n-1\right\} = 1$) so we are done with the base case $n = 2$. In what follows we make the induction hypothesis that the formula applies for $n-1$. Divide the line into $m+1$ connected intervals, $I_0, I_1, \dots, I_m$, where $m = \left\lfloor \frac{L}{b}\right\rfloor$. Interval $I_0$ starts at 0 and otherwise interval $I_j$ starts at $L-(m+1-j)b$. The leftmost point, which can be chosen in $n$ different ways, may be located in any of the $m+1$ intervals and we shall construct $V_D(n,b,L)$ as the sum of $m+1$ contributions:

$$V_D(n,b,L) = \sum_{j=0}^{m} V_{D,j}(n,b,L)$$

where $m = \left\lfloor \frac{L}{b}\right\rfloor$ and where the index $j$ tells in what interval the leftmost point is located. Integrating over starting position we have in general that

$$V_{D,j}(n,b,L) = n\left(\int_{\min I_j}^{\max I_j} f_j(x)dx - \int_{\min I_j}^{\max I_j} g_j(x)dx\right)$$

where from the induction hypothesis:

$$f_j(x) = \sum_{k=0}^{\min\{m-j,n-2\}} \nu_k\,(L-x-kb)^{n-1}$$

and

$$g_j(x) = \sum_{k=0}^{\min\{m-j-1,n-2\}} \nu_k\,(L-x-(k+1)b)^{n-1} = \sum_{k=1}^{1+\min\{m-j-1,n-2\}} \nu_{k-1}\,(L-x-kb)^{n-1}$$

with $\nu_k = (-1)^k\binom{n-2}{k}$. The subtraction of the integral with integrand $g_j(x)$ is to correct for the fact that the integral with integrand $f_j(x)$ may include volumes where the separation between the two leftmost points exceeds $b$. We get

$$V_{D,j}(n,b,L) = n\left(\int_{\min I_j}^{\max I_j} \nu_0(L-x)^{n-1}\,dx + \int_{\min I_j}^{\max I_j} \sum_{k=1}^{\min\{m-j,n-2\}} (\nu_k - \nu_{k-1})\,(L-x-kb)^{n-1}dx + \delta_1\right)$$

where

$$\delta_1 = \begin{cases} 0 & \text{if } m-j < n-2 \\ -\int_{\min I_j}^{\max I_j} \nu_{n-2}(L-x-(n-2)b)^{n-1}\,dx & \text{if } m-j \geq n-2 \end{cases}$$

Index $k$ appears in the evaluation of $V_{D,j}(n,b,L)$ if and only if $\min\{m-j,n-2\} \geq k$. Since the intervals are connected we can for any given $k$ merge the relevant integrals and get

$$V_D(n,b,L) = n\int_0^L \nu_0(L-x)^{n-1}\,dx + n\sum_{k=1}^{\min\{m-1,n-2\}} \int_0^{L-kb} (\nu_k - \nu_{k-1})(L-x-kb)^{n-1}\,dx + n\delta_2$$

where

$$\delta_2 = \begin{cases} 0 & \text{if } m-1 < n-2 \\ -\int_0^{L-(n-2)b} \nu_{n-2}(L-x-(n-2)b)^{n-1}\,dx & \text{if } m-1 \geq n-2 \end{cases}$$

Trivial calculus and incorporation of the aforementioned identities [$\nu_0 = \mu_0$, $-\nu_{n-2} = \mu_{n-1}$, $\nu_k - \nu_{k-1} = \mu_k$ for $1 \leq k \leq n-2$ and $m = \lfloor L/b \rfloor$] then gives

$$V_D(n,b,L) = \sum_{k=0}^{\min\{\lfloor L/b \rfloor,n-1\}} \mu_k(L-kb)^n$$

as desired.

**ENTRY 11:** *A problem involving logarithms*

This problem proposal, in which multiple questions are squeezed into single ones, was simply not deemed interesting enough.

Problem 11.1: Define

$$f(c,m,n,u) = \log_{c+1}\left(\frac{n+(c+1)^u}{m}\right) - u$$

and accept argument values only from the set of strictly positive integers, $\mathbb{Z}_+$, when treating the questions that follow.

(a) In what ways can the four arguments $c$, $m$, $n$ and $u$ be split into two groups, $A$ and $B$, of two arguments each, so that no matter what values are set for the arguments in $A$ it remains possible to assign values to the arguments in $B$ so that $f(c,m,n,u) \in \mathbb{Z}_+$?

(b) Provided that only three of the equations $c_1 = c_2, m_1 = m_2, n_1 = n_2$ and $u_1 = u_2$ are true, when can it happen that both $f(c_1, m_1, n_1, u_1)$ and $f(c_2, m_2, n_2, u_2)$ are in $\mathbb{Z}_+$?

*Solution*: (a) The pairs to which values can be set arbitrarily are limited to $(c,m), (c,u)$ and $(m,u)$. For fixed $c$ and $m$ we may set $n = (c+1)^2 m - (c+1)$ and $u = 1$ to get

$$f(c,m,(c+1)^2 m - (c+1),1) = \log_{c+1}\left(\frac{(c+1)^2 m - (c+1) + (c+1)^1}{m}\right) - 1$$

$$= \log_{c+1}((c+1)^2) - 1 = 2 - 1 = 1 \in \mathbb{Z}_+.$$

For fixed $c$ and $u$ we may set $m = 1$ and $n = (c+1)^{u+1} - (c+1)^u$ to get

$$f(c,1,(c+1)^{u+1} - (c+1)^u, u) = \log_{c+1}\left(\frac{(c+1)^{u+1} - (c+1)^u + (c+1)^u}{1}\right) - u$$

$$= \log_{c+1}((c+1)^{u+1}) - u = u + 1 - u = 1 \in \mathbb{Z}_+.$$

For fixed $m$ and $u$ we may set $c = 1$ and $n = 2^{u+1}m - 2^u$ to get

$$f(1,m,2^{u+1}m - 2^u, u) = \log_2\left(\frac{2^{u+1}m - 2^u + 2^u}{m}\right) - u$$

$$= \log_2(2^{u+1}) - u = u + 1 - u = 1 \in \mathbb{Z}_+.$$

Let us introduce $C = c + 1 \geq 2$ and note that a general requirement for $f(c,m,n,u) \in \mathbb{Z}_+$ is that there exists an integer $s > u$ such that

$$\frac{n + C^u}{m} = C^s$$

This enforces $n = mC^s - C^u = C(mC^{s-1} - C^{u-1})$ where the factor within parentheses surely exceeds 1. We see that $n$ in general must be a multiple of $C \geq 2$ in order for there to be a chance that $f(c,m,n,u) \in \mathbb{Z}_+$. Thus if $n$ happens to be arbitrarily selected as equal to a prime number there is in fact no way that $f(c,m,n,u) \in \mathbb{Z}_+$.

b) It can happen when $m_1 \neq m_2$ (for instance is $f(1,1,6,1) = 2$ and $f(1,2,6,1) = 1$ both members of $\mathbb{Z}_+$). It can also happen when $n_1 \neq n_2$ (for instance is $f(1,1,6,1) = 2$ and $f(1,1,14,1) = 3$ both members of $\mathbb{Z}_+$). It can also happen when $c_1 \neq c_2$ (for instance is $f(1,1,30,1) = 4$ and $f(5,1,30,1) = 1$ both members of $\mathbb{Z}_+$). Finally, it cannot happen when $u_1 \neq u_2$.

*Proof* that it cannot happen when $u_1 \neq u_2$: Set again $C = c + 1 \geq 2$. Note that the base $C$ representation of $C^a - C^b$ for positive integers $a > b \geq 1$ contains $a$ "digits", the $b$ last being zeros and the rest being $C - 1$: s (e.g., nines when working in base 10 and ones when working in base 2). Let $m = m_1 = m_2$, $n = n_1 = n_2$ and $c = c_1 = c_2$. Without loss of generality, assume that $u_1 > u_2$. For both $f(c,m,n,u_1)$ and $f(c,m,n,u_2)$ to be in $\mathbb{Z}_+$ requires the existence of integers $s_1 > u_1$ and $s_2 > u_2$ such that

$$\frac{n + C^{u_1}}{m} = C^{s_1}; \frac{n + C^{u_2}}{m} = C^{s_2}$$

where clearly we must have $s_1 > s_2$ following the assumption that $u_1 > u_2$. If $m$ is of the form $m = C^k$ with $k$ a non-negative integer we get that

$$n = C^{s_1+k} - C^{u_1}; n = C^{s_2+k} - C^{u_2}$$

The first equation is saying that $n$ has $s_1 + k$ digits in its base-$C$ represenation, while the other is saying that $n$ has $s_2 + k$ digits in its base-$C$ represenation. Since $n$ has a unique base-$C$ represenation it follows that $s_1$ and $s_2$ are forced to be equal, contradicting the notion $s_1 > s_2$ above. If $m$ is such that $C^k < m < C^{k+1}$ for some non-negative integer $k$ we get when picturing two solutions, with similar settings as above ($u_1 < s_1, u_2 < s_2, u_1 > u_2, s_1 > s_2$):

$$C^{k+s_1} - C^{u_1} < n < C^{k+1+s_1} - C^{u_1}$$

$$C^{k+s_2} - C^{u_2} < n < C^{k+1+s_2} - C^{u_2}$$

The inequalities tell that the number of digits in the base-$C$ represenation of $n$ on the one hand is $k + s_1$ or $k + 1 + s_1$ and on the other hand is $k + s_2$ or $k + 1 + s_2$. With $s_1 > s_2$ we can only realize this by setting $s_1 = s_2 + 1$. Using this we are forced to accept

$$C\frac{n + C^{u_2}}{m} = C^{s_2+1} = C^{s_1} = \frac{n + C^{u_1}}{m}$$

meaning that we have to accept that $n = (C^{u_1} - C^{u_2+1})/(C - 1)$. But then we are accepting that $n$ has at most $u_1$ digits, contradicting that $n$ has $k + s_1$ or $k + 1 + s_1$ digits since $u_1 < s_1$.

**ENTRY 12:** *An ABC-problem*

A version of this problem was considered to rely too heavily on numerical calculations.

Problem 12.1: Let $k \geq 1$ be a positive integer. The numbers $1, 2, \dots, 3k + 1$ will be randomly permuted as to generate a vector $v$. You get only to see $v_1, \dots, v_k$ and then be informed whether or not it holds true that at least one of the equalities $\sum_{i=k+1}^{2k} v_i = \prod_{j=2k+1}^{3k} v_j$ and $\sum_{i=2k+1}^{3k} v_i = \prod_{j=k+1}^{2k} v_j$ holds true. What is the probability, $P(k)$, that you will be able to figure out with certainty the value of $v_{3k+1}$?

*Solution*: The answer is $P(k) = 0$ for $k = 1$ and $k \geq 5$ while $P(2) = 4/315$, $P(3) = 1/175$ and $P(4) = 1/1925$. For $k = 1$ there is no way for the "sum to equal the product" as that would require two elements of the vector to have the same value. The maximum sum of $k$ elements is given by the expression $s_k = (5k^2 + 3k)/2$ and the minimum product is $p_k = k!$. Note that for $k \geq 5$ we get

$$p_k - s_k = k\left((k-1)! - \frac{5}{2}k - \frac{3}{2}\right) \geq k\left(6(k-1) - \frac{5}{2}k - \frac{3}{2}\right)$$
$$= k\left(\frac{7}{2}k - \frac{15}{2}\right) = \frac{k}{2}(7k - 15) > 0$$

so also for $k \geq 5$ there is no way for the "sum to equal the product".

For $k = 2{,}3$ and 4 we calculate $P(k)$ through

$$P(k) = \frac{T_k}{Z_k}$$

Here $Z_k$ is the numbers of ways to distribute the $3k + 1$ elements into three hands (A, B, C) containing $k$ elements (labelled cards) each and one hand containing a single element. $T_k$ is the number of ways for A, B and C to be such that "sum B" = "product C" and/or "product B" = "sum C" and knowledge of that allows to work out the value of the element in the single element pile from the elements seen in hand A. By the combinatorial multiplication principle

$$Z_k = \binom{3k+1}{k}\binom{2k+1}{k}\binom{k+1}{k} = \frac{(3k+1)!}{k!^3}$$

giving $Z_2 = 630$, $Z_2 = 16800$ and $Z_4 = 450450$. The computationally more challenging part brings (if done correctly) $T_2 = 8$, $T_3 = 96$ and $T_4 = 234$. The procedure is essentially the following (with clarification for the case with $k = 2$):

(i) Find all $k$-tuples with a product not exceeding $s_k$.

For $k = 2$ we have $s_2 = 13$ and the 2-tuples of interest are (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4)

(ii) For each of these $k$-tuples find all complementary $k$-tuples having element sums equal to the element product of the original $k$-tuple. Make a list of unique ordered sequences.

(1, 2): none. (1, 3): none. (1, 4): none. (1, 5): yes (2, 3). (1, 6): yes (2, 4). (1, 7): yes (2, 5) and (3, 4). (2, 3): yes (1, 5). (2, 4): yes (1, 7) and (3, 5). (2, 5): yes (3, 7) and (4, 6). (2, 6): yes (5, 7). (3, 4): yes (5, 7).

This gives a shortlist "1235, 1246, 1257, 1347, 1235, 1247, 2345, 2357, 2456, 2567, 3457" with unique ordered sequences (note that $2\times3 = 1 + 5$ and that $2 + 3 = 1\times5$) "1235, 1246, 1247, 1257, 1347, 2345, 2357, 2456, 2567, 3457"

(iii) Look at the $k$ visible elements and the list and work out whether the missing element can be figured out (requires one and only one sequence not containing the elements already visible).

(1, 2). **Yes**. A "yes" implies that 3457 can be formed by the elements in B and C so that the missing element must be 6. (1, 3). No. A "yes" leaves two options as there are two sequences missing both 1 and 3. (1, 4). No. Same. (1, 5). No. There is in fact no way that this renders a "yes". (1, 6). No. (1, 7). No.
(2, 3). No. (2, 4). No. (2, 5). **Yes**. Missing element must be 6. (2, 6). No. (2, 7). No.
(3,4). No. (3, 5) No. (3,6). No. (3, 7). No.
(4, 5). No. (4, 6) No. (4, 7) **Yes**. Missing element must be 6.
(5, 6). No. (5,7) **Yes.** Missing element must be 3.
(6, 7). No.

Multiply the number of **Yes** - occurrences by 2 to get $T_k$ (think of why!).

*Exercise*: Set up a code to verify $T_3 = 96$ and/or $T_4 = 234$.

**ENTRY 13:** *An infimum problem*

This one combines an infimum problem with an integral evaluation.

Problem 13.1: For non-negative real numbers $x, y$ and $z$ define $f(x, y, z)$ as the infimum of the expression $1/(x + a) + 1/(y + b) + 1/(z + c)$ over all non-negative real numbers $a$, $b$ and $c$ with $a + b + c = 1$. Evaluate $\int_0^1 \int_0^1 \int_0^1 f(x, y, z) dxdydz$.

*Solution*: Let $I = \int_0^1 \int_0^1 \int_0^1 f(x, y, z) dxdydz$. We shall show that

$$I = \frac{3}{4}(160 \ln 2 - 90 \ln 3 - 7)$$

To begin with, let us note that if $0 < \varepsilon < q_1 < q_2$ are real numbers then follows:

$$\frac{1}{q_1 + \varepsilon} + \frac{1}{q_2} = \frac{q_1 + q_2 + \varepsilon}{q_1 q_2 + q_2 \varepsilon} < \frac{q_1 + q_2 + \varepsilon}{q_1 q_2 + q_1 \varepsilon} = \frac{1}{q_1} + \frac{1}{q_2 + \varepsilon}$$

which goes to show that one should always add to the smallest denominator when seeking to minimize the sum of reciprocals with equal numerators. It follows that if it is possible to make the denominators equal to each other, then so should be done.

Let us assume that $0 \le x \le y \le z \le 1$ and introduce variables $p_1, p_2$ and $p_3$ to aid in the construction of $a$, $b$ and $c$. We first let $p_1 = y - x$. After this step we have that $x + p_1 = y \le z$. If $2(z - y) \le (1 - p_1)$ we can set $p_2 = z - y$ and get that $x + p_1 + p_2 = y + p_2 = z$. We can then distribute the rest (what remains to reach the sum 1) equally as to get $x + p_1 + p_2 + p_3 = y + p_2 + p_3 = z + p_3$ with $p_3 = 1 - p_1 - 2p_2$. In this case we get $x + p_1 + p_2 + p_3 = y + p_2 + p_3 = z + p_3 = (1 + x + y + z)/3$ so that $f(x, y, z) = 9/(1 + x + y + z)$. If instead $2(z - y) > (1 - p_1)$ we must construct $p_2$ differently in view of the fact that we now have $(1 - p_1)/2 = (1 + x - y)/2$. In this case $x + p_1 + p_2 = y + p_2 = (1 + x + y)/2 < z$ and consequently we get $f(x, y, z) = 4/(1 + x + y) + 1/z$.

We shall evaluate $I$ as $I = J + 6K - 6L$ where

$$J = \int_0^1 \int_0^1 \int_0^1 \frac{9dxdydz}{1 + x + y + z}$$

$$K = \int_0^{1/2} \int_x^{1-x} \int_y^{\frac{1+x+y}{2}} \left(\frac{4}{1 + x + y} + \frac{1}{z}\right) dzdydx$$

$$L = \int_0^{1/2} \int_x^{1-x} \int_y^{\frac{1+x+y}{2}} \left(\frac{9}{1 + x + y + z}\right) dzdydx$$

Each integral can be evaluated in Mathematica and full simplification of the sum leads to the answer given above. The "sixes" in $I = J + 6K - 6L$ are there to cover the six possible "rankings" of $x, y$ and $z$. The limits of $K$ and $L$ are specific for a ranking with $x \leq y \leq z$. First is to be noted that if $x > 1/2$ there is no way that $2z - 2y > 1 + x - y$ since then would the equivalent inequality $2z - y > 1 + x$ be composed of a left hand side with a maximum value of $3/2$ ($z$ is at most equal to 1, $y$ is in this case at least equal to $1/2$ since $y \geq x$) and a right hand side with a minimum value of $3/2$. Next, with $x \leq 1/2$ and with $y > 1 - x$ we can also show that there is no way for $2z - y > 1 + x$. Let $y = 1 - x + \varepsilon$ where $\varepsilon$ is a small positive number. Then the inequality becomes $2z - (1 - x + \varepsilon) > 1 + x$ which can be rewritten as $2z > 1 + x + (1 - x + \varepsilon) = 2 + \varepsilon$ which is impossible to have fulfilled since $z \leq 1$. Finally, note that $2z - y > 1 + x$ is equivalent with $z > (1 + x + y)/2$. Hence, to avoid that $2z - y > 1 + x$ we must have that $z \leq (1 + x + y)/2$.

**ENTRY 14:** *A sequence enumeration problem*

A referee suspected that similar mappings as the one in the problem below may be covered in text books on computations. Again, if a reader is aware of earlier works dealing with the mapping in question I would like to know so that I can add references as a remark. For clarity, note that any positive integer has a unique *factorial base representation*. For instance, $17 = 2\times3! + 2\times2! + 1\times1!$ and the factorial base representation of 17 can then be said to be [2 2 1].

Problem 14.1: Let $V$ be the set of all finite sequences of nonnegative integers with a strictly positive initial element. Is there a bijective mapping from $V$ to the set of positive integers such that if $v \in V$ pairs with the positive integer $n$, then the sum of the elements in $v$ equals the sum of the coefficients in the factorial base representation of $n$?

*Solution*: The answer is yes. For $m = 2$ express 1 as {2,1}, for $m = 3$ express 2, 3, 4 and 5 as {2,1,3}, {2,3,1}, {3,1,2} and {3,2,1}, respectively. To express 6 to 23 in a similar way we need to proceed to $m = 4$ with e.g., 6 expressed as {2,1,3,4}, 17 as {3,4,2,1} and 23 as {4,3,2,1}. In general we express the number $n$ by identifying the integer $m$ such that $(m - 1)! \leq n < m!$ From this point we have that $(m - 1)!$ is expressed by {2,1,3,4,…, $m$} while $n$ then corresponds to some "higher" permutation of the same elements. Let us take the representation of $n$ as given by the vector $\boldsymbol{b} = \{b_1, b_2, \dots, b_m\}$ where $b_1, b_2, \dots, b_m$ are pairwise distinct integers $\leq m$ and $\geq 1$ and where $b_1 \geq 2$. On the one hand we can compute a vector of length $m-1$ by "asking" each of $b_1$, $b_2$,…, $b_{m-1}$ how many smaller elements that are present to its right. On the other hand we can compute a vector of length $m-1$ also by "asking" each of $b_2$, $b_3$,…, $b_m$ how many greater elements that are present to its left. Clearly the sum of the answers will be equal in the two cases since each counting "lower element to the right" mirrors in a counting "greater element to the left". The first computed vector corresponds neatly to the factorial base representation of $n$ while the second corresponds to some proper integer sequence if omitting all (potentially existing) initial zeros. That the mapping is bijective can be understood from that we can apply the procedure starting from any arbitrary sequence of $V$. As an example, the sequence $v =$ [2 0 5 3 3 8 5] has 8 at its sixth position. We may insert two zeros to shift the 8 to the 8: th position. From a modified $v$ vector of

$$v = [0\ 0\ 2\ 0\ 5\ 3\ 3\ 8\ 5]$$

we get

$$\boldsymbol{b} = \{3, 8, 9, 4, 10, 2, 6, 7, 1, 5\}$$

and a factorial base representation of [2 6 6 2 5 1 2 2 0] corresponding to $n = 10^6$.

**ENTRY 15:** *A problem inspired by Kaprekar's constant*

Consider the following algorithm:

i) Start with any four-digit number including ones with leading zeros but excluding ones where all four digits are the same.
ii) Construct the largest and the smallest numbers containing all four digits and calculate the difference.
iii) Using the difference, repeat step (ii) until arriving at a difference seen before.

As an example, say that we start with 7665: 7665 → 7665-5667 = 1998 → 9981-1899 = 8082 → 8820 – 0288 = 8532 → 8532-2358 = 6174 → 7641 – 1467 = 6174 (stop!)

No matter what acceptable four-digit number one starts with the outcome will be that the procedure ends at 6174. This is famously known as Kaprekar's constant. One may say that

$$(d-a-c)10^3 + (c-b-a)10^2 + (b-c-d)10 + (a-d-b) = 0$$

is only satisfied by $(a, b, c, d) = (1,4,6,7)$ under prevalent restrictions. Replacing the $10^k$:s by other expressions and seeking to solve the resulting equation with different restrictions led to the following problem proposal, the rejection of which in part may have been due to me not finding a smooth way to illustrate the (perhaps vague) connection to Kaprekar's constant.

Problem 15.1: Let $a, b, c, d$ and $t$ be integers with $0 \leq a < b < c < d$, $t \geq 1$ and with

$$(d-a-c)\left(\frac{abcd}{t}\right)^3 + (c-b-a)\left(\frac{abcd}{t}\right)^2 + (b-c-d)\left(\frac{abcd}{t}\right) + (a-d-b) = 0$$

We refer by a *solution* to a quintuple $(a, b, c, d, t)$ fulfilling the above requirements.

(a) Are there infinitely many solutions?
(b) Is there any solution with $t$ being non-composite?
(c) Is there any solution with $t$ being semiprime?

*Solution*: We shall arrive at the following answers:

(a) Yes, for instance, for any positive integer $n$ one may select to set $a = 5n, b = 10n + 2$, $c = 15n + 3, d = 20n + 3$ and $t = 6n(5n + 1)(20n + 3)$ to get a solution.

(b) No, there is no solution with $t$ being equal to 1 or equal to a prime number.

(c) Yes, there are solutions with semiprime $t$, e.g., $a = 1, b = 2, c = 3, d = 7$ and $t = 21$.

Introduce $M = \frac{abcd}{t}$. Assume first that $M$ is an integer such that $M > d$. Then we can rewrite the equation as

$$(d - a)M^3 + (c - b)M^2 + (b - c)M + (a - d) = cM^3 + aM^2 + dM + b$$

and apply $M-$adic subtraction (recall $0 \leq a < b < c < d < M$) to get an equation system with the four equations: $b = M + a - d, d = b - 1 - c + M, a = c - 1 - b$ and $c = d - a$. For a fixed value of $a$ we are left with equally many unknowns as equations and we find a unique solution with:

$$M = 5(a + 1), b = 2(a + 1), c = 3(a + 1), d = 4a + 3. \qquad (*)$$

For consistency we require then that

$$\frac{abcd}{t} = \frac{a[2(a+1)][3(a+1)](4a+3)}{t} = 5(a + 1) = M$$

$$\Leftrightarrow 6a(a + 1)(4a + 3) = 5t \qquad (**)$$

The left hand-side contains at least four (not necessarily distinct) prime factors so for a solution to exist while $M$ is an integer with $M > d$ cannot $t$ be restricted to have at most two prime factors. The provided answer to part (a) follows by setting $a = 5n$ in (*) and (**).

To complete part (b) we must rule out the existence of solutions with non-composite $t$ when $M \leq d$ and/or when $M$ is not an integer. Multiplying the equation in the problem formulation by $t^3$ and rearranging gives the equation

$$(d - a - c)(abcd)^3 + (c - b - a)t(abcd)^2 + (b - c - d)t^2(abcd) - dt^3 = (b - a)t^3$$

Note that the left hand side has $d$ as integer factor. We must then have that there exist a positive integer $k$ such that $kd = (b - a)t^3$. Since $d > (b - a)$ follows that $t^3 > k$. On the one hand this immediately rules out the possibility that $t = 1$. On the other hand, if $t$ is prime follows that the prime factorization of $k$ cannot contain three factors equal to $t$ and hence must the prime factorization of $d$ contain at least one factor equal to $t$. This shows that a hypothetical solution when $t$ is prime must be such that $d/t \geq 1$ and such that $M$ is an integer.

Note that the equation in the problem formulation also can be rearranged as:

$$d(M^3 - M - 1) = a(M^3 + M^2 - 1) + b(M^2 - M + 1) + c(M^3 - M^2 + M)$$

This can be rewritten as

$$d = af_a(M) + bf_b(M) + cf_c(M)$$

where the functions $f_a(x)$, $f_b(x)$ and $f_c(x)$ are defined by

$$f_a(x) = \frac{x^3+x^2-1}{x^3-x-1}; f_b(x) = \frac{x^2-x+1}{x^3-x-1}; f_c(x) = \frac{x^3-x^2+x}{x^3-x-1}$$

Since $abc \geq 6$ and since $d/t \geq 1$ (when $t$ is prime) follows that $M = abcd/t \geq 6$. When $x \geq 6$ it is easy to show that $1 < f_a(x) \leq 251/209$, $0 < f_b(x) \leq 31/209$ and $186/209 \leq f_c(x) < 1$. It so follows that (note that $a \geq 1$, $b \geq 2$ and $c \geq 3$):

$$d = af_a(M) + bf_b(M) + cf_c(M) < \frac{251}{209}a + \frac{31}{209}b + c < \frac{bc}{3}a + \frac{ac}{6}b + \frac{ab}{2}c = abc \leq \frac{abcd}{t} = M$$

or in short $d < M$. But we have already excluded solutions where $t$ is a prime number when $M$ is an integer exceeding $d$.

For part (c). Numerical calculations restricting $t$ and $d$ to values $\leq 120$ render five solutions

$a = 1, b = 2, c = 3, d = 7, t = 21$
$a = 1, b = 3, c = 5, d = 7, t = 35$
$a = 1, b = 6, c = 9, d = 10, t = 72$
$a = 1, b = 3, c = 5, d = 10, t = 75$
$a = 1, b = 2, c = 8, d = 13, t = 104$

of which the first two listed involve semiprime values of $t$. From the quintuple on the third line may also be noted that solutions must not be such that $abcd/t$ is integer valued.

**ENTRY 16:** *A problem involving primes and Pythagorean triangles*

A Pythagorean triangle is a right angled triangle with exclusively integer side lengths.

Problem 16.1: Verify that the following conversation actually makes sense:

Alice: - "Hmm. So what were you saying?"
Bob: - "I'm saying that $n$ is a positive even number…"
Alice: - "And what were you saying about $S_n$?"
Bob: - "That it only contains the numbers $\frac{n}{2} + 1, \frac{n}{2} + 2$ and so on until $n - 2$."
Alice: - "Ok. And you were saying that there is only one element, say $k$, in $S_n$, with the property that $n - k + 1$ is a proper divisor of $k$?"
Bob: - "Yes, and also that there is no element, say $t$, within $S_n$ being lower than $k$ and such that $n + 1 - t$ is a proper divisor of $2t - n$."
Alice: - "So you are saying that $n/2$ is any even leg of a Pythagorean triangle in which the other leg and the hypotenuse are primes?"
Bob: - "Yes, exactly!"

*Solution*: Let $r$ be an integer such that for some $k \in S_n$

$$r = \frac{k}{n-k+1} = \frac{1}{\frac{n+1}{k} - 1} \Leftrightarrow \frac{n+1}{k} = \frac{1+r}{r} \Leftrightarrow k = \frac{r(n+1)}{r+1}$$

We get that $n+1$ is of the form $m(r+1)$ while $k$ then is of the form $mr$. The existence of another solution $k_1$ would require the possibility to express $n+1 = m_1(r_1+1)$ and moreover $k_1 = m_1 r_1$. We would have $mr + m = m_1 r_1 + m_1$ at the same time as $mr \neq m_1 r_1$.

If $n+1$ would be prime than we would be forced to set $m = 1$. But then we get $r = n+1$ and $k = (n+1)^2/(n+2)$ which clearly is not an integer since $n+1$ and $n+2$ have no common prime factors. If $n+1 = pq$ with $p$ an odd prime number and $q > 1$ an odd number we could set $m = p$ to get $r = q-1$ and $k = \frac{(q-1)(n+1)}{q} = \frac{(q-1)pq}{q} = (q-1)p \in \mathbb{Z}$. We could alternatively set $m_1 = q$ to get $r_1 = p-1$ and $k_1 = \frac{(p-1)(n+1)}{p} = \frac{(p-1)pq}{p} = (p-1)q \in \mathbb{Z}$. It can be shown that $k \in S_n$ and $k_1 \in S_n$, we shall only verify it for $k$ here (the other is shown analogously) . Note that

$$k = (q-1)p = pq - p = n+1-p \leq n-2$$

and since $p > 2$

$$k = pq - q > pq - \frac{pq}{2} = \frac{pq}{2} = \frac{n+1}{2} \Rightarrow k \leq \frac{n}{2} + 1$$

It seems as if the only way to get a single element $k$ with the property that $n-k+1$ is a proper divisor of $k$ is for $n+1$ to be of the form $p^2$. We then get $k = p(p-1)$ which obviously is part of $S_n$. Thus we now know that we are required to have $n+1 = p^2$.

Next we seek to explore requirements on $n$ as to exclude

$$\frac{2t-n}{n+1-t} \in \mathbb{Z}$$

while $\frac{n}{2} + 1 \leq t < k$. Note that $n+1 = p^2$ so that $n = (p-1)(p+1)$ is divisible by 4. This shows that $n+2$ is of the form $2p_1q_1$ where $p_1$ and $q_1$ are odd and $p_1$ is a prime. We may set $2q_1 = Q_1$. We have that $n+2 = p^2+1 = 2p_1q_1 = p_1Q_1$ meaning that either $p_1$ or $Q_1$ exceeds $p$. Let $H = \max(p_1, Q_1)$ and $L = \min(p_1, Q_1)$. Select $t = HL - (H+1)$ as to get

$$\frac{2t-n}{n+1-t} = \frac{2(t+1)-(n+2)}{n+2-(t+1)} = \frac{2(HL-H)-HL}{HL-(HL-H)} = L-2$$

Only $L = 2$, i.e., $q_1 = 1$, can prevent this from being an integer. We return to this case soon but work for the moment with the assumption that $L > 2$. Then *does* $t = HL - (H+1)$ bring a positive integer ratio and moreover since $H > p$

$$HL - (H+1) = n+2-H-1 = n+1-H = p^2 - H < p(p-1) = k \Longrightarrow t < k$$

and since we are working with $L > 2$

$$HL - (H+1) = n + 1 - H = n + 1 - \frac{n+2}{L} > n + 1 - \frac{n+2}{2} = \frac{n}{2} \Rightarrow t \le \frac{n}{2} + 1$$

If $L = 2$ we get the scenario that $\frac{n}{2} + 1$ is a prime number and we seek positive integers $t$ and $s$ so that

$$\frac{2t - n}{n + 1 - t} = s \Leftrightarrow t + 1 = \frac{2\left(\frac{n}{2} + 1\right)(s + 1)}{s + 2}$$

The equivalence follows from non-displayed algebra. Note that $s + 1$ and $s + 2$ have no common prime factors so for $t$ to be an integer must $s + 2 = 2\left(\frac{n}{2} + 1\right)$ or $s + 2 = \frac{n}{2} + 1$. Both of these cases (as can be seen by re-expressing $s + 1$ accordingly) forces $t > p(p-1) = k$. As we do not want solutions we require $\frac{n}{2} + 1$ to be prime. In net we require $n + 1 = p^2$ and $\frac{n}{2} + 1 = q$ with $p$ and $q$ odd prime numbers. Note that with this setting $p^2 - q^2 = n + 1 - \left(\frac{n}{2} + 1\right)^2 = \left(\frac{n}{2}\right)^2$ so $\frac{n}{2}$ fits indeed as an even leg in a Pythagorean triangle whose other leg and hypotenuse are both prime.

We conclude by remarking on the (possibly well known) fact that in any Pythagorean triangle with an even leg and with the other sides prime, the hypotenuse must be exactly 1 unit length longer than the even leg. With $p$ and $q$ prime with $p > q$ and with $1 \le m \le p - 1$ we get

$$p^2 - q^2 = (p - m)^2 = p^2 - 2mp + m^2 \Leftrightarrow q^2 = m(2p - m)$$

forcing us to set $m = 1$ since $m = q$ would enforce $p = q$ and since $m = q^2$ would force us to set $2p - q^2 = 1$ which is not possible since $p - m \ge 1$ and hence $2p - m > 1$.

**ENTRY 17:** *A laser-and-mirrors type of problem*

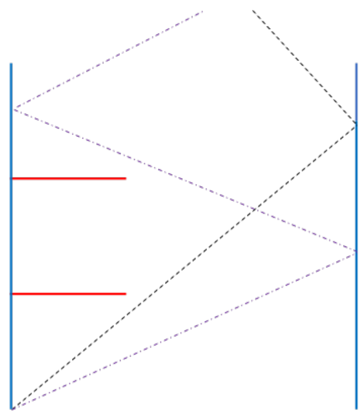
This one was considered somewhat tedious. The referee saw ways to shorten the solution.

Problem 17.1: In the Cartesian plane are drawn blue solid lines between (0,0) and (0,1) and between (1,0) and (1,1). For each positive integer, $p$, strictly smaller than the positive integer $m$, is then drawn a red line line between $(0, p/m)$ and $(1/m, p/m)$. A moving point is then ejected northeast from (0,0) and subject to no forces. If hitting a blue line it is reflected with the angle of incidence equal to the angle of reflection.

(a) For what combinations of $m$ and $n$ is it possible that the moving point makes its first contact with the line between (0,1) and (1,1) having touched no red line and having been reflected exactly $n$ times?

(b) Consider the same setup specifically for $m = 5$ but with a northwest ejection from (1,0) of the moving point. Show that there exist red-line-avoiding ways out that involve exactly $n$ reflections if and only if the final digit of $n$ is 0, 3, 6 or 9.

The figure below shows the setup with $m = 3$ as well as two examples of trajectories starting from (0,0) and avoiding red lines. One trajectory involves $n = 1$ reflection, the other $n = 2$ reflections.



*Solution*: (a) For $m = 1$ any finite value of $n$ works, including $n = 0$. For $m \geq 2$, any strictly positive value of $n$ works as long at it is not of the form $2km$ or $2km - 1$ where $k$ is a positive integer.

In the case with $m = 1$ there is no red line at all to worry about and we can adjust the ejection angle of the moving point to make any number of reflections possible. We assume in what follows that $m \geq 2$. As the angle of incidence equals the angle of reflection we can, instead of considering reflections, let the moving point continue along a straight line to ever growing $x$ values. For this analogy to work we must, however, place red lines not only between $(0, p/m)$ and $(1/m, p/m)$ for $p = 1,2 \dots, m - 1$ but also, for any positive integer $k$, between $(2k - 1/m, p/m)$ and $(2k + 1/m, p/m)$. Any straight line with positive slope that goes from $(0,0)$ to $(x = x_* > 0,1)$ without touching a red line signals a "good $m, n$ combination" if in addition the integer part of $x_*$ equals $n$. It is easy to see that the the red line between $(2k - 1/m, 1/m)$ and $(2k + 1/m, 1/m)$ when viewed from $(0,0)$ shadows out the region with $y = 1$ spanning $x$ values from $x'$ to $x''$ where $x' = 1/(1/m/(2k - 1/m)) = 2km - 1$ and $x'' = 1/(1/m/(2k + 1/m)) = 2km + 1$. All $x$ values strictly within the interval are characterized by integer parts of either $2km - 1$ or $2km$ and, as in shadow, follows that a combination of $m \geq 2$ with $n$ of the form $2km$ or $2km - 1$ (where $k$ is a positive integer) cannot "be good". Note that while casting shadows (at $y = 1$) over the entire intervals of $x$-values characterized by integer parts of $2km - 1$ and $2km$, the red lines characterized by $p = 1$ cast no shadow on any part of any other integer interval.

We shall prove that the red lines with $p > 1$ cannot shadow out the entirety of any "integer interval". Let us make a few observations. First, a red line at height $y = p/m$ casts a shadow at $y = 1$ of width $2/p$, since:

$$1/\left(\frac{p}{m}/\left(2k + \frac{1}{m}\right)\right) - 1/\left(\frac{p}{m}/\left(2k - \frac{1}{m}\right)\right) = \frac{2}{p}$$

Second, a red line with $p = 2$ casts a shadow at $y = 1$ centered around an integer value, since:

$$1/\left(\frac{2}{m}/(2k)\right) = km$$

Third, the shadow centers at $y = 1$ due to two adjacent red lines with the same value of $p$ are separated by more than 2 length units since the separation must exceed the separation between the centers of the adjacent red lines which is exactly 2 length units. The observations tell that an integer interval at $y = 1$ at most can be 50% shaded by a red line with $p = 2$. Moreover, if an integer interval is partly shaded by a red line with $p = 2$, it cannot be partly shaded by any other red line with $p = 2$. It follows also that the same integer interval cannot be partly shaded by more than one red line of any fixed $p$ value. We shall now show that a collection of red lines of pairwise distinct $p > 1$ values cannot shadow out the entirety of an integer interval. Note that such a scenario would require there to be shadows overlapping that do not possess common shadow centers. On the contrary we shall prove that if it happens that a point at $y = 1$ is in the shadow of two distinct red lines then must hold true that the projected shadow of each have a common center. That is, if there exists a real number $x$, such that there exists distinct positive integers $k'$ and $k''$ and distinct positive integers $p' < p'' < m$ with

$$\begin{cases} \dfrac{2k'm - 1}{p'} < x < \dfrac{2k'm + 1}{p'} \\ \dfrac{2k''m - 1}{p''} < x < \dfrac{2k''m + 1}{p''} \end{cases}$$

then must be the case that $\frac{2k'm}{p'} = \frac{2k''m}{p''}$. The statement just made is equivalent to the following lemma the proof of which completes our solution of part (a).

*Lemma* 1: Given that $a, b, m, p$ and $q$ are positive integers with $p < q < m$, then exists a real valued $x$ with $\frac{2am-1}{p} < x < \frac{2am+1}{p}$ and $\frac{2bm-1}{q} < x < \frac{2bm+1}{q}$ if and only if $a/p = b/q$.

*Proof of Lemma 1*: Assume first that $\frac{a}{p} = \frac{b}{q} = r$. Then the first interval goes from $2rm - \frac{1}{p}$ to $2rm + \frac{1}{p}$ while the other interval goes from $2rm - \frac{1}{q}$ to $2rm + \frac{1}{q}$. It is clear that both intervals contain $x = 2rm$ and hence is the intersection of the intervals non-empty. Assume in what follows that $\frac{a}{p} \neq \frac{b}{q}$.

We shall begin by showing that when $\frac{a}{p} \neq \frac{b}{q}$ and when $p < q < m$ it follows that $\frac{2am-1}{p} < \frac{2bm-1}{q}$ implies that $\frac{2am+1}{p} < \frac{2bm-1}{q}$. If we can show that $\frac{2bm-1}{q} - \frac{2am-1}{p} > \frac{2}{p}$ we are done since obviously the difference $\frac{2am+1}{p} - \frac{2am-1}{p} = \frac{2}{p}$. We have that

$$\frac{2bm - 1}{q} - \frac{2am - 1}{p} = \frac{1 - \frac{p}{q} + \frac{2m}{q}(bp - aq)}{p}$$

Note that $0 < 1 - \frac{p}{q} < 1$ and that $\frac{2m}{q} > 2$. It must be that $bp - aq$ equals an integer. The integer in question cannot be zero since $bp = aq \Leftrightarrow \frac{b}{q} = \frac{a}{p}$ which contradicts $\frac{a}{p} \neq \frac{b}{q}$. If $bp - aq$ is a negative integer then would follow that $\frac{1}{p}\left(1 - \frac{p}{q} + 2m\left(\frac{bp}{q} - a\right)\right) < \frac{1+2(-1)}{p} < 0$ contradicting the very fact that $\frac{2bm-1}{q} - \frac{2am-1}{p} > 0$. We must thus assume that $bp - aq$ is a positive integer and we then get that

$$\frac{2bm-1}{q}-\frac{2am-1}{p}=\frac{1-\frac{p}{q}+\frac{2m}{q}(bp-aq)}{p}>\frac{0+2\times(1)}{p}=\frac{2}{p}$$

We shall finally show that when $\frac{a}{p}\neq\frac{b}{q}$ and when $p<q<m$ it follows that $\frac{2am-1}{p}>\frac{2bm-1}{q}$ implies that $\frac{2am-1}{p}>\frac{2bm+1}{q}$. If we can show that $\frac{2am-1}{p}-\frac{2bm-1}{q}>\frac{2}{q}$ we are done since obviously the difference $\frac{2bm+1}{q}-\frac{2bm-1}{q}=\frac{2}{q}$. We have that

$$\frac{2am-1}{p}-\frac{2bm-1}{q}=\frac{1}{q}-\frac{1}{p}+\frac{2m}{pq}(aq-bp)$$

and since $p<q$ must the integer $aq-bp$ be positive in order for $\frac{2am-1}{p}-\frac{2bm-1}{q}$ to exceed zero. Since $aq-bp\geq 1$ follows that

$$\frac{1}{q}-\frac{1}{p}+\frac{2m}{pq}(aq-bp)\geq\frac{1}{q}-\frac{1}{p}+\frac{2}{q}\frac{m}{p}=\frac{2}{q}\left(\frac{m}{p}+\frac{1}{2}-\frac{q}{2p}\right)$$

It remains to show that $\frac{m}{p}+\frac{1}{2}-\frac{q}{2p}>1$. But since $q\leq m-1$ and since $m+1>p$ follows that

$$\frac{m}{p}+\frac{1}{2}-\frac{q}{2p}\geq\frac{m}{p}+\frac{1}{2}-\frac{m-1}{2p}=\frac{2m-m+1}{2p}+\frac{1}{2}=\frac{m+1}{2p}+\frac{1}{2}>\frac{p}{2p}+\frac{1}{2}=1.$$

(b) We leave this as an exercise, merely noting that the scenario is equivalent to one with northeast ejection from (0,0) and with red lines shifted to instead be attached to the right wall.

**ENTRY 18:** *Problems solved by turning sums into integrals*

I suspect that the trick utilized in the solution to the problem below is not original. A multitude of special series identities (including the one in the exercise at the end of the entry) can be worked out via the same type of step.

Problem 18.1: Evaluate

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt{k}\left(\sqrt{n-k+4}-\sqrt{n-k}\right)$$

*Solution*: The answer is $\pi$. We have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt{k}\left(\sqrt{n-k+4}-\sqrt{n-k}\right)\quad=\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{k}{n}}\left(\sqrt{\frac{n-k+4}{n}}-\sqrt{\frac{n-k}{n}}\right)$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{k}{n}}\left(\frac{\sqrt{\frac{n-k+4}{n}}-\sqrt{\frac{n-k}{n}}}{4/n}\right)\frac{4}{n} \quad = 4\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{k}{n}}\left(\frac{\sqrt{\frac{n-k+4}{n}}-\sqrt{\frac{n-k}{n}}}{4/n}\right)\frac{1}{n}$$

$$= 4\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{k}{n}}\left(\frac{f\left(1-\frac{k}{n}+h_*\right)-f\left(1-\frac{k}{n}\right)}{h_*}\right)\frac{1}{n}$$

where we have introduced $h_* = 4/n$. But since $h_* \to 0$ as $n \to \infty$ can the expression in the big parentheses be viewed as the derivative of $f(x)$ if setting $f(x) = \sqrt{x}$ and $x = 1 - \frac{k}{n}$. Then is $\frac{k}{n} = 1 - x$ and we may view $\sqrt{k/n}$ as $g(x) = \sqrt{1-x}$. Stepping $k$ from 1 to $n$, each time accounting for the factor $1/n$, may be viewed as taking care of the $dx$ while integrating from 0 to 1. We thus get that

$$4\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{k}{n}}\left(\frac{f\left(1-\frac{k}{n}+h_*\right)-f\left(1-\frac{k}{n}\right)}{h_*}\right)\frac{1}{n} = 4\int_0^1 g(x)f'(x)dx$$

$$= 4\int_0^1 \frac{\sqrt{1-x}}{2\sqrt{x}}dx = 2\int_0^1 \sqrt{\frac{1}{x}-1}\,dx = \pi$$

where $f(x) = \sqrt{x}$ , $f'(x) = \frac{1}{2\sqrt{x}}$ , $g(x) = \sqrt{1-x}$ and where the equality $\int_0^1 \sqrt{\frac{1}{x}-1}\,dx = \frac{\pi}{2}$ is considered well known.

*Exercise*: Prove that

$$\lim_{n\to\infty}\sum_{k=1}^{n-1}\ln\left(\frac{k}{n}\right)\ln\left(\frac{n-k}{n+1-k}\right) = \frac{\pi^2}{6}$$

**ENTRY 19:** *Cutting papers and breaking sticks problems*

The following problem contains several steps passed by via the use of Mathematica. The solution contains two implicit exercises, namely to reason why the integrands of the double integrals look the way they do.

Problem 19.1: A point is selected randomly on the perimeter of a rectangular paper. Another point is then selected randomly from one of the other three sides. A straight cut is made between the two points so that the paper is divided into two regions with areas $A_1 \leq A_2$. Let $\varepsilon$ be the expectation value of $A_1/A_2$. Determine the range of possible values of $\varepsilon$.

*Solution*: We shall show that $\varepsilon$ can only take any value fulfilling $\varepsilon_1 \leq \varepsilon < \varepsilon_2$ where

$$\varepsilon_1 = \frac{\ln 2^{48} - 42 + 2\pi^2 - 12(\ln 2)^2}{18} \approx 0.291379\ldots;\ \varepsilon_2 = \ln 2^8 - 5 \approx 0.545166\ldots$$

For points on opposite sides we get (rectangular area elements transformed to squares):

$$\varepsilon_{opp} = 2\int_{x_1=0}^{1}\int_{x_2=1-x_1}^{1}\frac{1-\frac{x_1+x_2}{2}}{\frac{x_1+x_2}{2}}dx_2dx_1 = 2\int_{x_1=0}^{1}\int_{x_2=1-x_1}^{1}\frac{2-(x_1+x_2)}{x_1+x_2}dx_2dx_1 = \ln 2^8 - 5$$

where Mathematica was used for the final step. For points on adjacent sides we get:

$$\varepsilon_{adj} = \int_{x_1=0}^{1}\int_{x_2=0}^{1}\frac{x_1x_2/2}{1-x_1x_2/2}dx_2dx_1 = \int_{x_1=0}^{1}\int_{x_2=0}^{1}\frac{x_1x_2}{2-x_1x_2}dx_2dx_1 = \frac{\pi^2}{6} - (1+(\ln 2)^2)$$

where Mathematica was used for the final step. For an aspect ratio of $a{:}b$ follows that

$$\varepsilon = \left[\left(\frac{a}{a+b}\right)\left(\frac{a}{a+2b}\right)+\left(\frac{b}{a+b}\right)\left(\frac{b}{2a+b}\right)\right]\varepsilon_{opp} + \left[\left(\frac{a}{a+b}\right)\left(\frac{2b}{a+2b}\right)+\left(\frac{b}{a+b}\right)\left(\frac{2a}{2a+b}\right)\right]\varepsilon_{adj}$$

As $\frac{b}{a} \to \infty$ (and, by symmetry, as $\frac{b}{a} \to 0$) we get $\varepsilon \to 1\varepsilon_{opp} + 0\varepsilon_{adj} = \varepsilon_{opp}$. As $\varepsilon_{opp} > \varepsilon_{adj}$ is there no way for $\varepsilon$ to reach or exceed $\varepsilon_{opp} = \varepsilon_2$, but one can come arbitrarily close. To minimize $\varepsilon$ we seek to make the factor in front of $\varepsilon_{adj}$ as big as possible. We introduce $x = b/a$ to get

$$\left(\frac{a}{a+b}\right)\left(\frac{2b}{a+2b}\right)+\left(\frac{b}{a+b}\right)\left(\frac{2a}{2a+b}\right) = \left(\frac{1}{1+x}\right)\left(\frac{2x}{1+2x}\right)+\left(\frac{x}{1+x}\right)\left(\frac{2}{2+x}\right) \equiv f(x)$$

and find through Mathematica (or other way) that $f(x)$ has its global maximum at $x = 1$ with $f(1) = \frac{2}{3}$. We thus get that $\varepsilon \geq \frac{1}{3}\varepsilon_{opp} + \frac{2}{3}\varepsilon_{adj} = \varepsilon_1$ and we leave out presenting the trivial steps needed to reach the closed form of $\varepsilon_1$ presented above.

*Exercise*: A thin stick is broken on two random locations into three smaller sticks of lengths $s_1 \leq s_2 \leq s_3$. Show that the expectation value of $s_1/s_3$ is given by $1 + \ln\sqrt{t}$ where $t = \frac{2^{12}}{3^9}$.

**ENTRY 20:** *A sequence within a sequence problem*

Before tackling the problem below take as an exercise to prove that $b_1, b_2, \ldots$ (as introduced in the problem statement) really exist.

Problem 20.1: Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of strictly increasing positive integers (e.g., the prime numbers). Let $b_0 = 0$ and let, for integers $k \geq 1$, $b_k$ be the minimum nonnegative integer that makes $(a_k + b_{k-1} - b_k)/a_{k+1}$ an integer Let $j_m$ be the index of the $m$:th entry among $b_0, b_1, b_2, \ldots$ that is strictly higher than its lower-index neighboring element.

(a) Is $j_1, j_2, j_3, \ldots$ an infinitely long sequence?

(b) Show that $b_{j_m} \geq a_{j_m}$ and that $b_{j_{m+1}} - b_{j_m} = a_{j_m+1}$.

*Solution*: a) Yes, it is. Assume the opposite, that $j_1, j_2, j_3, \ldots$ is not an infinitely long sequence. Then is there a maximum index $k = k^*$ for which $b_k > b_{k-1}$. For $k > k^*$ we must then always have $b_{k-1} - b_k < 0$ since $b_{k-1} = b_k$ would imply

$$\frac{a_k + b_{k-1} - b_k}{a_{k+1}} = \frac{a_k}{a_{k+1}}$$

which cannot be an integer since with $0 < a_k < a_{k+1}$ we get $0 < a_k/a_{k+1} < 1$. But since $b_{k^*}$ is finite is it impossible to continue in limitless fashion reducing the value of subsequent "$b$-values" while respecting that each $b$-value must be a nonnegative integer.

b) We shall let $s = j_m$ and $s + n = j_{m+1}$. We have that $a_{s+1} > a_s$ and $b_s > b_{s-1}$ so the fact that

$$\frac{a_s + b_{s-1} - b_s}{a_{s+1}} \in \{0,1,2, \ldots\}$$

implies that $a_s + b_{s-1} = b_s$ which immediately tells that $b_s \geq a_s$ since $b_{s-1} \geq 0$. It remains to show that $b_{s+n} - b_s = a_{s+1}$. We do this via a lemma.

*Lemma 1:* It holds true that $b_k < a_{k+1}$ for all $k \geq 1$ and that $b_{k+1}$ is given by the minimum nonnegative number among $a_{k+1} + b_k - a_{k+2}$ and $a_{k+1} + b_k$.

*Proof by induction*: We always have $b_1 = a_1$ so that $b_1 < a_2$. Assume that $b_k < a_{k+1}$ for all integers up to $k$. We shall show that this implies that $b_{k+1} < a_{k+2}$. We have that

$$\frac{a_{k+1} + b_k - b_{k+1}}{a_{k+2}} < \frac{2a_{k+1} - b_{k+1}}{a_{k+2}} < 2 - \frac{b_{k+1}}{a_{k+2}}$$

and so is

$$\frac{a_{k+1} + b_k - b_{k+1}}{a_{k+2}} \in \{0, 1\}$$

We get that $b_{k+1} = a_{k+1} + b_k - a_{k+2}$ if the expression to the right is nonnegative and that $b_{k+1} = a_{k+1} + b_k$ otherwise. In the latter case we directly get $a_{k+2} > b_{k+1}$ while in the former case we may introduce $c = a_{k+2} - a_{k+1}$ to get that $b_{k+1} = (a_{k+2} - c) + b_k - a_{k+2} = b_k - c$ so that $b_{k+1} < b_k < a_{k+1} < a_{k+2}$. ***

Note in particular in the proof of the lemma that $b_{k+1} = a_{k+1} + b_k - a_{k+2}$ applies to cases where $b_{k+1} < b_k$. If instead $b_{k+1} > b_k$ then applies $b_{k+1} = a_{k+1} + b_k$. It thus follows that

$$b_{s+1} = a_{s+1} + b_s - a_{s+2}$$
$$b_{s+2} = a_{s+2} + b_{s+1} - a_{s+3} = a_{s+2} + (a_{s+1} + b_s - a_{s+2}) - a_{s+3} = a_{s+1} + b_s - a_{s+3}$$
$$b_{s+3} = a_{s+3} + b_{s+2} - a_{s+4} = a_{s+1} + b_s - a_{s+4}$$
$$\ldots$$
$$b_{s+n-1} = a_{s+n-1} + b_{s+n-2} - a_{s+n} = a_{s+1} + b_s - a_{s+n}$$

and finally (note that $b_{s+n} > b_{s+n-1}$)

$$b_{s+n} = a_{s+n} + b_{s+n-1} = (a_{s+1} + b_s - b_{s+n-1}) + b_{s+n-1} = a_{s+1} + b_s$$

so that $b_{s+n} - b_s = a_{s+1}$, as desired.