C9. Douglas Lind; suggested by editors. Show that there are infinitely many numbers that appear at least six times in Pascal's triangle.

Solution. For $m \geq 3$, m occurs twice as $\binom{m}{1}$ and $\binom{m}{m-1}$. By symmetry, it will suffice to find infinitely many values of m with at least two more occurrences in the left half of the triangle.

There are several small examples of such pairs of occurrences: $120 = \binom{10}{3} = \binom{16}{2}$, $210 = \binom{10}{4} = \binom{21}{2}$, $1540 = \binom{22}{3} = \binom{56}{2}$, and $3003 = \binom{15}{5} = \binom{14}{6}$. The last of these exhibits the intriguing relationship $\binom{n}{k} = \binom{n-1}{k+1}$. To solve the problem, we will find infinitely many solutions of this equation with $k > 1$ and $k + 1 < (n-1)/2$.

The equation $\binom{n}{k} = \binom{n-1}{k+1}$ is equivalent to $n(k+1) - (n-k)(n-k-1) = 0$. We claim that for every positive integer j, this equation is satisfied by the values $n = F_{2j+2}F_{2j+3}$ and $k = F_{2j}F_{2j+3}$, where F_i is the ith Fibonacci number. To see why, note that with these values we have $n - k = (F_{2j+2} - F_{2j})F_{2j+3} = F_{2j+1}F_{2j+3}$, and therefore

$$\begin{aligned} n(k+1) - (n-k)(n-k-1) &= F_{2j+2}F_{2j+3}(F_{2j}F_{2j+3} + 1) - F_{2j+1}F_{2j+3}(F_{2j+1}F_{2j+3} - 1) \\ &= F_{2j+3}(F_{2j+2}F_{2j}F_{2j+3} + F_{2j+2} - F_{2j+1}^2F_{2j+3} + F_{2j+1}) \\ &= F_{2j+3}(F_{2j+2}F_{2j}F_{2j+3} - F_{2j+1}^2F_{2j+3} + F_{2j+3}) \\ &= F_{2j+3}^2(F_{2j+2}F_{2j} - F_{2j+1}^2 + 1) = 0, \end{aligned}$$

where the last step uses the well-known identity $F_{i+1}F_{i-1} - F_i^2 = (-1)^i$.

The case $j = 1$ yields $n = 15$ and $k = 5$, the example we found earlier. When $j = 2$ we get $n = 104$ and $k = 39$, and indeed $\binom{104}{39} = \binom{103}{40} = 61218182743304701891431482520$.

Editorial comments. The appearance of the Fibonacci numbers in this solution can be explained by reference to classic problem C2 (this MONTHLY, Feb. 2022, p. 194). Viewing the equation $n(k+1) - (n-k)(n-k-1) = 0$ as a quadratic in n and applying the quadratic formula yields

$$n = \frac{3k + 2 \pm \sqrt{5k^2 + 8k + 4}}{2}.$$

For n to be an integer, we need $5k^2 + 8k + 4$ to be a perfect square. Setting $5k^2 + 8k + 4 = t^2$ and solving for k by the quadratic formula, we get

$$k = \frac{-4 \pm \sqrt{5t^2 - 4}}{5}.$$

For k to be an integer, $5t^2 - 4$ must be a perfect square, and the solution to classic problem C2 (March 2022, pp. 293–294) shows that this happens if and only if t is an odd-indexed Fibonacci number. Setting $t = F_{2i+1}$ and applying Fibonacci identities leads to the values

$$n = F_{i+1}F_{i+2} + \frac{(-1)^{i+1} - 1}{5}, \quad k = F_{i-1}F_{i+2} + \frac{4((-1)^{i+1} - 1)}{5}.$$

These are integers when i is odd, and setting $i = 2j + 1$ leads to the values used in the solution.

This result is due to Lind [1; see also 3, 4]. It is related to a 1971 conjecture of Singmaster [2; see also 6]. For an integer m with $m \geq 2$, let S_m be the number of times m appears in Pascal's triangle. Singmaster conjectured that S_m is bounded, and suggested that 10 or

12 might be a bound. The problem shows that 5 cannot be an asymptotic bound. It turns out that $S_{3003} = 8$; there are no other known values of m for which $S_m \geq 8$. The sequence of binomial coefficients for which $S_m \geq 6$ starts 120, 210, 1540, 3003, 7140, 11628, 24310, 61218182743304701891431482520 (see [5]).

References.

1. D. Lind, The quadratic field $Q(\sqrt{5})$ and a certain Diophantine equation, *Fib. Quart.* **6** (1968) 86–94, `https://www.fq.math.ca/Scanned/6-3/lind.pdf`.

2. D. Singmaster, How often does an integer occur as a binomial coefficient?, *Amer. Math. Monthly* **78** (1971) 385–386.

3. D. Singmaster, Repeated binomial coefficients and Fibonacci numbers, *Fib. Quart.* **13** (1975) 295–298, `https://www.fq.math.ca/Issues/13-4.pdf`.

4. C. A. Tovey, Multiple occurrences of binomial coefficients, *Fib. Quart.* **23** (1985) 356–358.

5. OEIS sequences: `https://oeis.org/A003015`, `https://oeis.org/A003016`, `https://oeis.org/A090162`.

6. K. Matomäki, M. Radziwiłł, X. Shao, T. Tao, and J. Teräväinen, Singmaster's conjecture in the interior of Pascal's triangle, `https://arxiv.org/abs/2106.03335`.