# Keeping Dry: The Mathematics of Running in the Rain

DANK HAILMAN
Jamaicarain University

BRUCE TORRENTS
Raindrop-Macon College*

It's really coming down! As you ease into the nearest free parking spot in front of the supermarket the feeling of dread builds—you're going to get wet. But your mathematical mind wonders, what can I do, caught as I am without an umbrella, to stay as dry as possible as I make a run across the parking lot to the supermarket door? Will I stay drier if I run faster? Is there an optimal speed that will minimize my exposure?

These questions are not new, but they do have a knack for stumping people. Indeed, back in 1972, the MAGAZINE published a piece on this subject [**3**], and the following year issued a corrected version [**10**], as the original was deeply flawed. This pattern was repeated in the meteorological community in the mid 1990s, when the journal *Weather* published one piece [**7**] in 1995, and offered a corrected version (by different authors) in 1997 [**9**]. Even the television program *Mythbusters* got it wrong (episode 1) and later offered the revised conclusion that running usually trumps walking (episode 38). The results of these studies were summarized neatly in limerick by Matthew Wright [**11**] in 1995:

*When caught in the rain without mac,*
*Walk as fast as the wind at your back,*
*But when the wind's in your face*
*The optimal pace*
*Is fast as your legs will make track.*

In 2002, however, Herb Bailey [**2**] pointed out that the limerick above is only partially correct. It is true that in the case of a head-wind one should travel as quickly as possible. But although one does indeed stay driest by traveling "as fast as the wind at your back" in the case of a *strong* tail-wind, if the tail-wind is sufficiently weak, running "as fast as your legs will make track" is better. In fact, Bailey's argument is simply a restatement of the corrected MAGAZINE piece of 1973, where the same observation was made.

All of these analyses use a rectangular solid to model our damp traveler. In this paper we will study the prospects for more well rounded individuals. Our results for ellipsoidal travelers, for example, show that indeed, shape matters! For such travelers we take further issue with Wright's limerick. Our model suggests that in the presence of a tail-wind, however weak, it is *always* beneficial to move faster than the "wind at your back." In fact, we feel compelled to offer the following advice:

*When you find yourself caught in the rain,*
*while walking exposed on a plane,*
*for greatest protection*
*move in the direction*
*revealed by a fair weather vane.*

*Moving swift as the wind we'll concede,*
*for a box shape is just the right speed.*
*But a soul who's more rounded*
*will end up less drownded*
*if the wind's pace he aims to exceed.*

*Please direct correspondence to Dan Kalman (kalman@american.edu), American University, Washington, DC 20016, and Bruce Torrence (btorrenc@rmc.edu), Randoolph-Macon College, Ashland, VA 23005.

## Getting wet

We begin with some assumptions. First, since the subject has forgotten to bring an umbrella, it is reasonable to assume masculine gender. Thus we sacrifice gender neutrality and refer to him accordingly. Next we ask, how exactly is our wandering mathematician going to get wet? We will assume that rain is falling uniformly with constant velocity (no gusts). We represent this velocity by the vector $\mathbf{v}_r$ whose vertical component is negative. The key idea is this: focus on the region occupied by all the raindrops that will strike the traveler during his trip. We call this the *rain region*. The amount of water striking the traveler will be in proportion to the measure of the rain region (area in two dimensions, volume in three). Accordingly, we adopt this geometric measure as an index of total wetness.

Suppose first that our hero is standing still. Then regardless of the shape of our traveler, for each point $P$ on his body that will be hit by a raindrop, we can draw a line segment of fixed length into space from $P$ in the direction $-\mathbf{v}_r$ and conclude that every raindrop that will strike $P$ in a certain time frame lies precisely on this segment. Hence the rain region is the generalized cylinder composed of the union of all such line segments (one for each point on his body that is exposed to the elements).

Now assume that our traveler moves at a constant speed $s > 0$ along a horizontal line, and adopt a distance measure so that he travels a total distance of one unit. We orient a Cartesian coordinate system in such a way that a reference point on our traveler starts at the origin and moves in the positive $x$ direction. Thus the mathematician's velocity vector is $\mathbf{v}_m = \langle s, 0 \rangle$ in a two-dimensional model, or $\mathbf{v}_m = \langle s, 0, 0 \rangle$ in three. He is exposed to the elements for a finite amount of time, specifically $1/s$. The rain region consists of all initial locations from which a raindrop can land on the mathematician. Let $Q$ be such a location, corresponding to a raindrop that will land at time $t$. Then it will strike the mathematician at the point $Q + \mathbf{v}_r t$. That point in turn has traveled with the mathematician from its original location $P = Q + \mathbf{v}_r t - \mathbf{v}_m t$. Thus for every exposed point $P$ on the mathematician at time 0, the point $P + (\mathbf{v}_m - \mathbf{v}_r)t$ is in the rain region for $0 \le t \le 1/s$. This shows that the rain region is made up of line segments parallel to the *apparent* rain vector $\mathbf{v} = \mathbf{v}_r - \mathbf{v}_m$, each terminating at an exposed point on the mathematician at time 0, and each of length $\|\mathbf{v}\|/s$. A two-dimensional rendering of this scenario is shown in FIGURE 1 for two different bodies (one rectangular, one elliptical) and three different walking speeds, all in the case of a moderate head-wind. The figures correctly suggest that in these conditions, regardless of the precise shape of his body, the faster he moves, the smaller the area of the rain region, and hence the drier he stays.

**Rectangular bodies** In the case of a two-dimensional rectangular body as shown in the first row of FIGURE 1, the total wetness measure is simply the sum of the areas of two parallelograms, and analysis is straightforward. In the case of a three-dimensional rectangular solid body, the total wetness measure is the sum of the volumes of three parallelepipeds. That is, for each of the three exposed faces of the body, one finds the volume of the parallelepiped containing the rain that will strike this face. This volume is the product of area of the face with the magnitude of the projection of the vector $\mathbf{v}/s$ onto a line orthogonal to the face. For details, we refer the reader to Bailey [**2**] (who gives an equivalent, although less geometric analysis).

One of the advantages of our geometric approach is that it is easy to visualize the extreme cases. For a two-dimensional traveler whose speed precisely matches that of a tail-wind, the apparent rain vector is vertical. That is, all the rain that will strike our traveler lies directly above his initial position. At the other extreme, if we imagine that our mathematician is moving infinitely fast, the apparent rain vector is horizontal, for

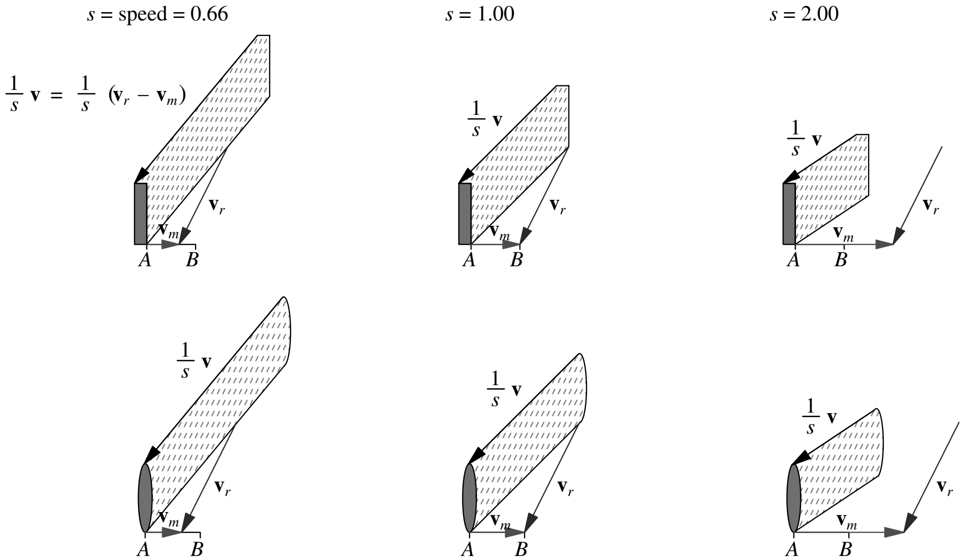


**Figure 1** Two different 2-dimensional bodies (yes, 2-dimensional—do not be misled into interpreting these as 3-dimensional figures) each moving at three different speeds. One is rectangular, one elliptical, and they travel under the same rain conditions (a moderate head-wind) from point *A* to point *B*, a total distance of one unit. Total wetness is measured as the area of the *rain region*, the region containing the rain that will strike the body.

all the rain that will strike him (or, more accurately, that he will strike) lies directly in front of him. Between these extremes, where our rectangular hero's speed is finite but exceeds that of any tail-wind, the area of the parallelogram containing the rain that will strike his *front* side is exactly that of the rectangle holding the rain that would strike his front side were he moving infinitely fast. But in this case the rain region also includes the parallelogram holding the rain that will strike his top side (and the area of this parallelogram diminishes as his travel speed increases). A practical conclusion is that in the absence of a tail-wind, a body stays driest by running as fast as possible.

However, in the case of a strong tail-wind (strong here is a relative term—in three dimensions it must be at about human walking speed in the absence of a cross-wind, but stronger if there is a cross-wind—see Bailey [**2**]), the optimal speed of travel for a cereal-box-shaped mathematician is *precisely* the speed of the tail-wind. It is easy to show the total wetness measure $T$ as a function of the speed $s$ of travel, has a critical point, though not necessarily a local minimum, at the speed $w$ of the tail-wind. FIGURE 2 illustrates this, showing the graphs of $T$ for various cross-wind values. Note that for all cross-wind values, the limit of $T$ as the travel speed $s \to \infty$ is simply the area of the front face of the body. This shows that all $T$ graphs share a common horizontal asymptote. When the cross-wind is sufficiently strong, the limiting value is a lower bound for the total wetness: the faster you go, the drier you stay. But for weaker cross-winds the $T$ curve approaches the asymptote from below. In this case, going too fast actually makes you wetter. A dynamic version appears at the MAGAZINE website.

**Other body shapes** Similar methods apply to bodies that are more complex polyhedral solids. For each exposed face of the body, one finds the volume of a generalized cylinder with the face as its base. Its volume is the area of this face times the magnitude of the projection of the vector $\mathbf{v}/s$ onto a line orthogonal to the face. Summing these volumes over all exposed faces gives a measure of the total amount of rain to strike

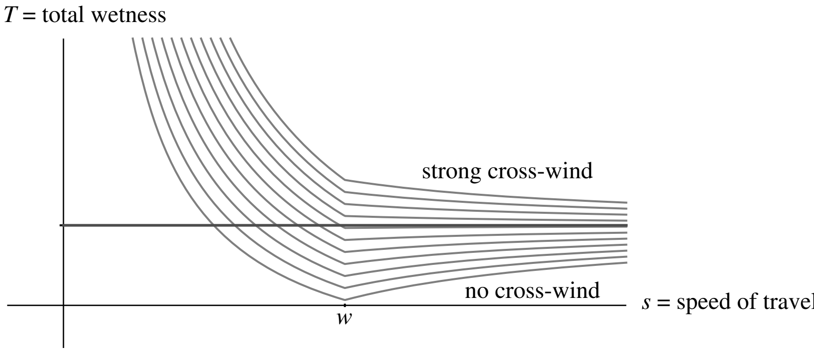


**Figure 2** Total wetness as a function of $s =$ speed of travel when traveling one distance unit. The speed $w$ of the tail-wind is constant. Several wetness functions are shown for various cross-wind speeds. All are asymptotic to the horizontal line $y = A$, where $A$ is the area of the front face of the traveler.

the body. For smooth surfaces, one could refine this approach into a surface integral. However, even for such simple surfaces as ellipsoids the resulting integral is difficult.

An alternate method that we pursue in the remainder of this work is to calculate the area of the projection of the body along the apparent rain vector $\mathbf{v}$ onto a plane orthogonal to $\mathbf{v}$. The volume of the rain region will be equivalent to the volume of the right cylinder whose base is this projection, and whose height is the magnitude $\|\mathbf{v}\|/s$, as illustrated in FIGURE 3 for an ellipsoidal body.

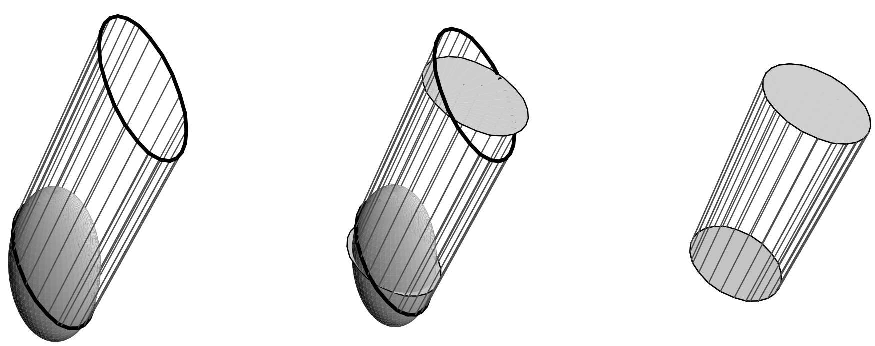

**Figure 3** Orthogonal projection of the body along the apparent rain vector to obtain a right cylinder of volume equal to the rain region.

**Spherical bodies** Few people are spherical in shape (although this is a widely accepted model for cows [**6**], and they may also wish to keep dry). Be that as it may, spherical bodies provide an irresistible temptation for modeling due to their symmetry. In the case of a spherical body, its orthogonal projection onto *any* plane through its center is always a disk of the same radius. In particular, variations in the apparent rain direction due to changes in the body's speed do not change the area of the projection. Take a spherical body of radius $r$, and once and for all let us fix the rain vector $\mathbf{v}_r = \langle w_t, w_c, -l\rangle$, so that a tail-wind is represented by a positive value for $w_t$, the cross-wind is represented by $w_c$, and $l > 0$ represents the downward speed of the rain. Using the projection approach outlined in FIGURE 3, our measure of total wetness is $\pi r^2$ times the magnitude of the vector $\mathbf{v}/s = (\mathbf{v}_r - \mathbf{v}_m)/s = \langle w_t - s, w_c, -l\rangle/s$. Thus we may write the total wetness function $T$ as

$$T(s) = \frac{\pi r^2 \sqrt{(w_t - s)^2 + w_c^2 + l^2}}{s}$$

It is easily verified that this function has a limiting value of $\pi r^2$ as $s \to \infty$, is strictly decreasing on $(0, \infty)$ when $w_t < 0$ (head-wind present), and that it has an absolute minimum at its lone critical point

$$s = \frac{w_t^2 + w_c^2 + l^2}{w_t} = \frac{\|\mathbf{v}_r\|^2}{w_t}$$

when $w_t > 0$ (tail-wind present). Thus, in contrast to the situation where the body was modeled with a rectangular solid, whenever there is a tail-wind, however weak, there is a particular speed $s$ at which a spherical body stays driest. Moreover, since the vertical component $l$ of the rain vector is nonzero, this speed is strictly *greater* than the speed $w_t$ of the tail-wind. This surprising state of affairs manifests itself not only for the sphere but for capsules and ellipsoids.

**R2-D2** It is a reasonably simple matter (at least in theory) to apply this analysis to a body that is a union of solids. Some shapes are particularly simple. Imagine, for instance, a capsule-shaped body composed of a right circular cylinder whose axis is parallel with the $z$ axis, capped above and below by a hemisphere of the same radius—a bit like the Star Wars droid R2-D2. An analysis like that for the sphere shows that if there is a tail-wind, however slight, there is a definite speed $s$ at which the body should move which minimizes the amount of rain to strike the body. As in the case of the sphere, this optimal speed is strictly greater than the speed of the tail-wind.

## Ellipsoidal projections

In order to generalize the above result to ellipsoidal bodies, it is necessary to calculate the area of the orthogonal projection of the ellipsoid along the apparent rain vector. This entire section is devoted solely to this endeavor; those readers whose primary interest is in the results for ellipsoidal travelers may safely skip ahead to the next section.

For nonspherical ellipsoids, the area of the orthogonal projection of the ellipsoid along the apparent rain vector will vary as the speed of travel (and hence the apparent rain vector) changes, and so things are considerably more complicated than with the case of the sphere. Nevertheless, there turns out to be a simple formula for the area of the projection. In fact, it generalizes beautifully to $n$ dimensions. Pursuing this more general approach, we unify the analysis for two and three-dimensional models. As a side benefit, we easily obtain results for an $n$-dimensional ellipsoidal mathematician dashing through the rain.

We consider an $n$-dimensional ellipsoid in $\mathbb{R}^n$, and will compute the $n-1$ dimensional measure of its projection on a hyperplane orthogonal to a given vector. Not surprisingly, matrices and determinants play a central role in the derivation. To begin, we prove a computational lemma that will be useful in the main argument. This result is a special case of a more general identity for the determinant of $A + B$ when $B$ is a rank one matrix. The general version, which is derived with a simple partitioned matrix argument in Meyer [**8**, p. 475], is equivalent to the Cauchy expansion of the determinant [**1**, pp. 74–75].

LEMMA. *For any collection $p_1, p_2, \ldots, p_n$ of nonzero real numbers, and any collection $r_1, r_2, \ldots, r_n$ of real numbers, the $n \times n$ matrix*

$$M = \begin{pmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_n \end{pmatrix} + \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} \begin{pmatrix} r_1 & r_2 & \cdots & r_n \end{pmatrix}$$

*has determinant $p_1 p_2 \cdots p_n \left(1 + \frac{r_1^2}{p_1} + \frac{r_2^2}{p_2} + \cdots + \frac{r_n^2}{p_n}\right)$.*

*Proof.* Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the standard basis vectors for $\mathbb{R}^n$, and let $\mathbf{r} = \langle r_1, r_2, \ldots, r_n \rangle$. Note that row $j$ of the matrix $M$ is the vector $p_j \mathbf{e}_j + r_j \mathbf{r}$. Since the determinant function is $n$-linear in the rows of $M$, and since each row of $M$ is itself a sum, we may express $\text{Det}(M)$ as an expansion, where each row in each matrix whose determinant appears in this expansion is either a multiple of a standard basis vector, or a multiple of $\mathbf{r}$. Note that in this expansion many terms are zero. In particular, the determinant of any matrix with two or more rows that are multiples of $\mathbf{r}$ is zero. Removing these terms, the expansion of the determinant of $M$ has the following form:

$$\text{Det} \begin{pmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_n \end{pmatrix} + r_1 \, \text{Det} \begin{pmatrix} r_1 & r_2 & \cdots & r_n \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_n \end{pmatrix}$$

$$+ r_2 \, \text{Det} \begin{pmatrix} p_1 & 0 & \cdots & 0 \\ r_1 & r_2 & \cdots & r_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_n \end{pmatrix} + \cdots + r_n \, \text{Det} \begin{pmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ r_1 & r_2 & \cdots & r_n \end{pmatrix}$$

and the result follows. ∎

Proceeding with our analysis, we characterize an ellipsoid in $\mathbb{R}^n$ using matrices. Given any real symmetric $n \times n$ matrix $M$ with positive determinant, we define the *generalized ellipsoid* for $M$ to be the set of (row) vectors $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle \in \mathbb{R}^n$ that satisfy the quadratic form equation $\mathbf{x} M \mathbf{x}^T = 1$. Note that there exists a real orthogonal matrix $P$ with the property that $PMP^{-1}$ is a diagonal matrix (with the same determinant as $M$). Note also that for any collection $a_1, \ldots, a_n$ of positive real numbers, the diagonal matrix

$$D = \begin{pmatrix} a_1^{-2} & 0 & \cdots & 0 \\ 0 & a_2^{-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^{-2} \end{pmatrix}$$

yields the standard generalized ellipsoid with equation $x_1^2/a_1^2 + x_2^2/a_2^2 + \cdots + x_n^2/a_n^2 = 1$. This ellipsoid has volume $U_n a_1 \cdots a_n$, where $U_n$ is the volume of the unit sphere in $\mathbb{R}^n$ (as can be seen from the fact that the linear transformation $x \mapsto \langle a_1 x_1, \ldots, a_n x_n \rangle$ with determinant $a_1 \cdots a_n$ maps the unit sphere onto this ellipsoid). Noting that for the diagonal matrix $D$ above we have $1/\sqrt{\text{Det} D} = a_1 \cdots a_n$, and that the ellipsoid associated with the matrix $P^{-1} D P$ has the same volume, we see that the ellipsoid associated with the matrix $M$ has volume $U_n / \sqrt{\text{Det} M}$.

Note that the quantity $U_n$ can be worked out explicity, and in fact is given by the equation $U_n = 2\pi^{n/2}/(n\Gamma(n/2))$. Although this is moderately involved, things simplify nicely by considering the even and odd cases separately. For example, in even dimensions we have $U_{2n} = \pi^n/n!$. Zatzkis [**12**] gives a complete derivation of this. Also, Fraser [**5**] and Dahlka [**4**] give clever derivations that do not make use of the gamma function.

Now let us turn to projections. For any nonzero $n$-dimensional vector $\mathbf{w}$, let $\pi_{\mathbf{w}} : \mathbb{R}^n \to \mathbb{R}^n$ be orthogonal projection along the vector $\mathbf{w}$ onto a hyperplane of dimension $(n-1)$.

THEOREM. *Let $\mathcal{E} \subset \mathbb{R}^n$ be the generalized ellipsoid defined by*

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} = 1,$$

*and let the vector $\mathbf{v} = \langle v_1, v_2, \ldots, v_n\rangle$ belong to $\mathcal{E}$. Then the projection $\pi_{\mathbf{v}}(\mathcal{E})$ of this ellipsoid has volume*

$$\frac{U_{n-1}}{\|\mathbf{v}\|} a_1 a_2 \cdots a_n.$$

*Proof.* Note that $\mathcal{E}$ is a level surface for the real-valued function $f(x_1, x_2, \ldots, x_n) = x_1^2/a_1^2 + x_2^2/a_2^2 + \cdots + x_n^2/a_n^2$. Hence the gradient $\nabla f$ when evaluated at a point of $\mathcal{E}$ is normal to $\mathcal{E}$ at that point. So the points of $\mathcal{E}$ satisfying the equation $\nabla f \cdot \mathbf{v} = 0$ are precisely those points for which $\mathbf{v}$ is a tangent vector. Collectively, these points of tangency form what we will call the *terminator ellipsoid* $E_T$. We note that $E_T$ is indeed an ellipsoid of dimension $n-1$, since the equation $\nabla f \cdot \mathbf{v} = (v_1/a_1^2)x_1 + (v_2/a_2^2)x_2 + \cdots + (v_n/a_n^2)x_n = 0$ is that of a hyperplane, and the intersection of any generalized ellipsoid in $\mathbb{R}^n$ and any $n-1$ dimensional subspace of $\mathbb{R}^n$ is always an ellipsoid of dimension $n-1$. In FIGURE 3 the terminator ellipse is drawn with a heavy black line in each of the first two frames.

We seek the volume of the projection $\pi_{\mathbf{v}}(\mathcal{E})$. Our first observation is that the boundary of this projection is $\pi_{\mathbf{v}}(E_T)$, and hence the volume we seek is that of the ellipsoid $\pi_{\mathbf{v}}(E_T)$. We denote this ellipsoid by $E$, and note that in FIGURE 3 the ellipse $E$ is drawn with a thin black line in the middle frame.

Consider again the hyperplane containing the terminator ellipsoid $E_T$. This hyperplane has normal vector $\mathbf{N} = \langle v_1/a_1^2, v_2/a_2^2, \ldots, v_n/a_n^2\rangle$. Since $\mathbf{v}$ belongs to $\mathcal{E}$, at least one $v_i \neq 0$, so we may suppose without loss of generality that $v_n \neq 0$. We solve for $x_n$ in the hyperplane equation:

$$x_n = -\frac{a_n^2}{v_n}\left(\frac{v_1}{a_1^2}x_1 + \frac{v_2}{a_2^2}x_2 + \cdots + \frac{v_{n-1}}{a_{n-1}^2}x_{n-1}\right).$$

One more ellipsoid is needed for our calculation: the vertical projection of the terminator ellipsoid, $\pi_{\langle 0,\ldots,0,1\rangle}(E_T)$, which we will call the *horizontal ellipsoid* $E_H$. It can be obtained by substituting the above expression for $x_n$ into the equation of the ellipsoid $\mathcal{E}$. Essentially we just eliminate the variable $x_n$ and obtain

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_{n-1}^2}{a_{n-1}^2} + \left(\frac{a_n}{v_n}\left(\frac{v_1}{a_1^2}x_1 + \frac{v_2}{a_2^2}x_2 + \cdots + \frac{v_{n-1}}{a_{n-1}^2}x_{n-1}\right)\right)^2 = 1 \qquad (1)$$

Writing $\mathbf{r} = (a_n/v_n)\langle v_1/a_1^2, \ldots, v_{n-1}/a_{n-1}^2\rangle$ and $\mathbf{x} = \langle x_1, \ldots, x_{n-1}\rangle$, and regarding vectors as matrices with one row, we have

$$\left(\frac{a_n}{v_n}\left(\frac{v_1}{a_1^2}x_1+\frac{v_2}{a_2^2}x_2+\cdots+\frac{v_{n-1}}{a_{n-1}^2}x_{n-1}\right)\right)^2=(\mathbf{x}\cdot\mathbf{r})^2=(\mathbf{x}\,\mathbf{r}^T)(\mathbf{x}\,\mathbf{r}^T)^T$$
$$=\mathbf{x}(\mathbf{r}^T\mathbf{r})\mathbf{x}^T$$

Thus we may express equation 1 for $E_H$ as the quadratic form equation

$$\mathbf{x}\left(\begin{pmatrix}\frac{1}{a_1^2} & 0 & \cdots & 0\\ 0 & \frac{1}{a_2^2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \frac{1}{a_{n-1}^2}\end{pmatrix}+\mathbf{r}^T\mathbf{r}\right)\mathbf{x}^T=1$$

Note that this matrix, $M$, has the form of the matrix in the lemma, and hence we have

$$\begin{aligned}\mathrm{Det}(M)&=\frac{1}{a_1^2\cdots a_{n-1}^2}\left(1+\left(\frac{a_n}{v_n}\right)^2\left(\frac{v_1^2}{a_1^2}+\cdots+\frac{v_{n-1}^2}{a_{n-1}^2}\right)\right)\\&=\frac{1}{a_1^2\cdots a_{n-1}^2}\left(1+\left(\frac{a_n}{v_n}\right)^2\left(1-\frac{v_n^2}{a_n^2}\right)\right)\\&=\frac{1}{a_1^2\cdots a_{n-1}^2}\left(\frac{a_n}{v_n}\right)^2\end{aligned}$$

We conclude that the volume of the horizontal ellipsoid $E_H$ is

$$\frac{U_{n-1}}{\sqrt{\mathrm{Det}(M)}}=U_{n-1}a_1a_2\cdots a_{n-1}\frac{|v_n|}{a_n}.$$

Knowledge of the volume of $E_H=\pi_{\langle 0,\ldots,0,1\rangle}(E_T)$, allows us to find our ultimate goal, the volume of ellipsoid $E=\pi_{\mathbf{v}}(E_T)$, for both are projections of $E_T$. The idea is that projecting a figure scales its measure by the cosine of a certain angle. This is most easily seen in $\mathbb{R}^3$, where projecting a figure in one plane orthogonally onto a second plane scales the area by the cosine of the dihedral angle between the two planes. This same idea works in $\mathbb{R}^n$. In particular we can relate the volumes of $E_T$, $E_H$, and $E$, using the known volume of $E_H$ to find the other two.

Since the projection $\pi_{\langle 0,\ldots,0,1\rangle}$ maps the terminator ellipsoid $E_T$ onto the horizontal ellipsoid $E_H$, we consider the angle between their respective hyperplanes. The acute angle between the horizontal hyperplane $x_n=0$ and the hyperplane of the terminator ellipsoid is the same as the angle between their normal vectors, provided this angle is acute. The normal vectors can be chosen as $\langle 0,\ldots,0,\pm 1\rangle$ and $\mathbf{N}=\langle v_1/a_1^2, v_2/a_2^2,\ldots,v_n/a_n^2\rangle$, where the sign in the first vector is chosen to match the sign of $v_n$, thus making the angle between them acute. Using the dot product, we find the cosine of this angle is

$$\frac{|v_n|}{a_n^2\,\|\mathbf{N}\|}.$$

Similarly, the projection $\pi_{\mathbf{v}}$ maps the terminator ellipsoid $E_T$ to the ellipsoid $E$. The acute angle between their respective hyperplanes is the angle between $\mathbf{N}$ and $\mathbf{v}$. Its

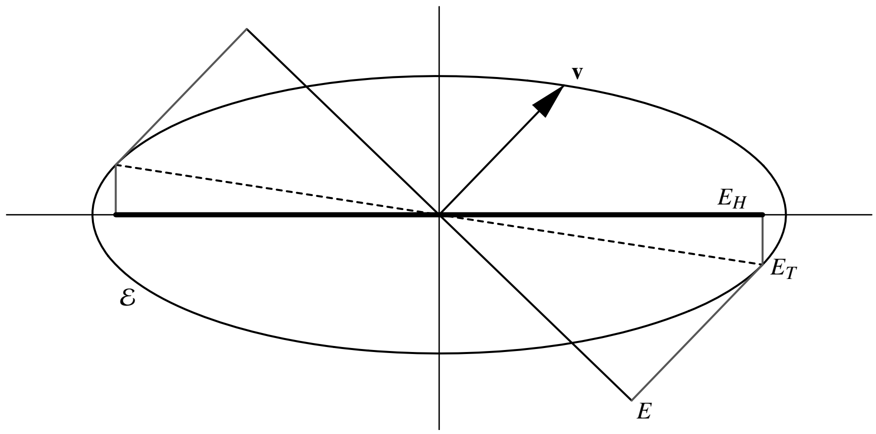


**Figure 4** In two dimensions, the ellipses $E_H$ and $E$ shown as projections of the terminator ellipse $E_T$.

cosine is given by

$$\frac{v_1^2/a_1^2 + v_2^2/a_2^2 + \cdots + v_n^2/a_n^2}{\|\mathbf{v}\|\,\|\mathbf{N}\|} = \frac{1}{\|\mathbf{v}\|\,\|\mathbf{N}\|}$$

since $\mathbf{v}$ is a member of $\mathcal{E}$. Putting this all together, we see that the volume of the ellipsoid $E$ is the volume of $E_H$ multiplied by the ratio of the cosines, which is

$$\left(U_{n-1} a_1 a_2 \cdots a_{n-1} \frac{|v_n|}{a_n}\right)\left(\frac{a_n^2}{|v_n|\,\|\mathbf{v}\|}\right) = \frac{U_{n-1}}{\|\mathbf{v}\|} a_1 a_2 \cdots a_n.$$

## Ellipsoidal bodies

Consider the ellipsoidal body $\mathcal{E}$ with equation $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, moving as before in the positive $x$ direction a distance of one unit with speed $s$, and with rain vector $\mathbf{v}_r = \langle w_t, w_c, -l\rangle$. The apparent rain vector is as before: $\mathbf{v} = \mathbf{v}_r - \mathbf{v}_m = \langle w_t - s, w_c, -l\rangle$. The measure of total wetness $T$ as a function of $s$ is the volume of the rain region, that is, the volume of the region containing the rain that will strike our ellipsoidal hero in the course of his journey. Specifically, it is the volume of the right cylinder whose base is the the projection $\pi_{\mathbf{v}}(\mathcal{E})$ and whose height is $\|\mathbf{v}\|/s$. To find the area of the base, we use our THEOREM, but take into account the fact that $\mathbf{v}$ may not lie on $\mathcal{E}$. Toward this end, choose $k > 0$ so that

$$k^2 = \frac{(w_t - s)^2}{a^2} + \frac{w_c^2}{b^2} + \frac{l^2}{c^2}.$$

Then $\mathbf{v}/k$ lies on $\mathcal{E}$. So by the theorem, the area of the projection $\pi_{\mathbf{v}}(\mathcal{E})$ is

$$\frac{U_2}{\|\mathbf{v}/k\|} a\,b\,c = \frac{k\,\pi}{\|\mathbf{v}\|} a\,b\,c$$

We multiply this area by the height of the cylinder to get volume

$$T(s) = \left(\frac{k\pi}{\|\mathbf{v}\|} a\,b\,c\right)\left(\frac{\|\mathbf{v}\|}{s}\right) = \frac{\pi}{s} k\,a\,b\,c \;=\; \frac{\pi}{s}\sqrt{k^2a^2b^2c^2}$$
$$= \frac{\pi}{s}\sqrt{b^2c^2(w_t - s)^2 + a^2c^2w_c^2 + a^2b^2l^2}.$$

This formula reduces to that found earlier for spherical bodies when $a = b = c$. It is readily verified that this total wetness function $T$ has a limiting value of $\pi bc$ (the area of the projection of the ellipsoid onto the $yz$ plane) as $s \to \infty$, is strictly decreasing on $(0, \infty)$ when $w_t \leq 0$ (no tail-wind), and that it has an absolute minimum at its lone critical point

$$s_{\text{opt}} = \frac{b^2c^2w_t^2 + a^2c^2w_c^2 + a^2b^2l^2}{b^2c^2w_t}$$

when $w_t > 0$ (tail-wind present). This optimal speed is again strictly *greater* than the speed $w_t$ of the tail-wind. Moreover, if the traveler becomes very skinny from back to front ($a \to 0$), the optimal speed approaches $w_t$.

For example, consider an ellipsoid of roughly human proportions, with $a = 1$, $b = 2$, and $c = 6$ (units are not important; it is only the relative dimensions that are relevant). And imagine rain conditions where the vertical downward rainfall speed is $l = 12$ mph, with a tail-wind $w_t = 5$ mph and a cross-wind $w_c = 5$ mph. In this case the total wetness measure $T(s)$ is minimized when the body moves at a speed of $s = 7.05$ mph, well above the speed of the tail-wind! (See the FIGURE 5, where the wetness at speeds $s = 5$ and 7.05 are highlighted).

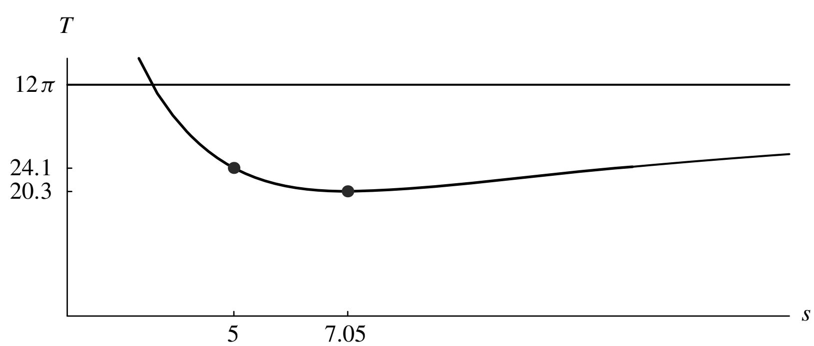


**Figure 5** With a 5 mph tail-wind, this elliptical body stays driest by traveling at approximately 7 mph. The MAGAZINE website hosts a dynamic version of the figure.

Moreover, it is definitely advantageous to the body to move at this speed rather than the speed of the tail-wind; in this example the body gets roughly 19% wetter when moving at the speed of the tail-wind instead of the optimal speed. This unexpected result is contrary to that predicted by the rectangular solid model, where total wetness is minimized when the body moves precisely at the speed of the tail-wind (provided the tail-wind is sufficiently strong, as it is in this example).

For an ellipsoidal traveler moving in tail-wind conditions, then, there are three travel speeds worthy of our attention: the speed $w_t$ of the tail-wind, the optimal speed $s_{\text{opt}}$, and the traveler's top running speed, which we will denote by $s_{\max}$. Suppose that conditions are such that $0 < w_t < s_{\max}$. We now investigate how much wetter a wandering ellipsoid can possibly get by traveling at the less-than-ideal speeds $w_t$ or $s_{\max}$ than he would get by proceeding at the optimal pace.

The example above suggests that one can get roughly 20% wetter by traveling at the speed $w_t$ of the tail-wind. Moreover, these are in conditions where the rectangular solid model recommends that the traveler move at precisely the speed of the tail-wind. Under the body dimensions considered above, and under the constraints that our hero walk no more slowly than human walking speed (say 3 mph) and that the weather conditions favor having a box-shaped body optimally travel at the speed of the tail-wind, this value of 20% is near the upper limit on how much wetter an ellipsoid will get by slowing to the speed of the tail wind versus traveling at the faster optimal pace.

Under these conditions, however, our mathematician would get only slightly wetter by running flat-out than he would by hitting the optimal tempo. In the previous example, for instance, running at 9 mph (a brisk pace to sustain in slippery conditions) gets him a little over 5% wetter than moving at the best pace. Will this always be the case, or are there atmospheric conditions when moving at the optimal pace keeps an ellipsoid *significantly* drier than running flat-out? To investigate this, consider the ratio

$$R = \frac{T(s_{\max})}{T(s_{\text{opt}})}.$$

This ratio measures how much wetter a running body will get than a body traveling at the optimal pace. In the case of either ellipsoidal or cereal-box-shaped travelers, it is a simple matter to deduce that this ratio is maximized when the cross-wind $w_c = 0$. In other words, the traveler is moving precisely in the direction of the wind. Equivalently, we need only consider a two-dimensional model. In the case of an elliptical traveler with semi-axes $a$ and $c$ moving when there is a tail-wind at speed $w$, it is straightforward to calculate

$$R = \frac{\sqrt{(a^2l^2 + c^2w^2)(a^2l^2 + c^2(s_{\max} - w)^2)}}{a\,c\,l\,s_{\max}}$$

Noting that the numerator is symmetric in $w$ and $s_{\max} - w$, it is a simple matter to deduce that $R$ attains its maximum precisely when

$$w = \frac{s_{\max}}{2}$$

where it attains a maximum value of

$$R_{\max} = \frac{al}{cs_{\max}} + \frac{cs_{\max}}{4al}.$$

In other words, the wind conditions under which an elliptical traveler will pay the highest price for running flat-out as opposed to moving at the optimal pace is when the tail-wind speed is exactly *half* the traveler's top running speed. A similar calculation for rectangular travelers was carried out by Schwartz and Deakin [**10**]. If one then substitutes the human-like values $a = 1$ and $c = 6$, and uses a top running speed $s_{\max} =$ 9 mph and a vertical rainfall velocity $l = 12$ mph, one finds that in the worst case (a 4.5 mph tail-wind), $R$ is approximately 1.34. That is, our elliptical traveler cannot get more than 34% wetter when running as opposed to traveling at the optimal pace. Traveling at the optimal pace, therefore, *can* keep him much drier than running. We note that the ratio $R$ is sensitive to changes in both $s_{\max}$ and $l$. The ratio will be even higher in light rain conditions (small $l$), and will be also be higher for faster runners.

Using these same relative dimensions in the case of a rectangular traveler, however, and using the formula for this ratio provided by Schwartz and Deakin [**10**], the maximal ratio is approximately 1.8. In both cases the body stays significantly drier by moving at the optimal pace appropriate for his body shape. But the penalty for running flat-out is greater for boxes than for ellipsoids!

## Keeping dry

Before addressing the subtleties suggested by these models, it is important to understand that all of this only applies to relatively short stints in the rain. One can only get so wet before being saturated, so that additional rain just runs off. (In this instance, the mathematician is said to be all wet.) Furthermore, if one walks in the rain when winds are light, or moves near the speed of a tail-wind, one's head takes most of the water (unless there is a calculus book on top of it), and eventually it will drip down to the face and body. Our models don't take this redistribution of water into account—again: short stints in the rain.

In the absence of a tail-wind, regardless of body shape, one stays driest by traveling as quickly as possible. With a tail-wind, the ellipsoidal model suggests traveling slightly *faster* than the speed of the tail-wind, as if to outrun the rain. This differs from the rectangular solid analysis, which suggests that the body move at the *same* speed as the tail-wind (or move as fast as possible if the cross-wind is sufficiently high or the tail-wind sufficiently weak). Our first conclusion, then, is simply that *shape matters*. Moreover, in the case of an ellipsoidal traveler, small perturbations to the lengths of the axes of the ellipsoid will change the optimal speed of travel (for rectangular solids in strong tail-wind conditions, the optimal pace is simply the speed of the tail-wind, and so is immune to any changes in the dimensions of the traveler). Again, the ellipsoidal model reveals the simple truth that shape matters.

But in practical terms, the models considered here suggest that conditions where traveling at some optimal speed (related to the speed of a tail-wind) will keep one *significantly* drier than running at full speed are rare. In particular, a tail-wind must exist but be about half of one's top running speed, and the cross-wind must be minimal for this effect to be apparent. However, in these ideal conditions both the rectangular and ellipsoidal models suggest that a traveler will stay significantly drier by moving at the optimal pace, especially in a light rain.

Our recommendation, therefore, is to *RUN* in the rain unless you find yourself traveling in the perfect storm—where the tail-wind is half your top running speed, the cross-wind is minimal, and the rainfall is light. In such conditions, given the rounded features of the human body, it might make sense to dampen your pace (so to speak) from a run down to a speed that is just a bit *faster* than that of the tail-wind.

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