The Quadrahelix:
A Nearly Perfect Loop of Tetrahedra

Michael Elgersma (Minneapolis, MN, USA); elgersma.michael@gmail.com
Stan Wagon (Macalester College, St. Paul, MN, USA); wagon@macalester.edu

Dedicated to the memory of Stash Świerczkowski (1932–2015), mathematician and adventurer.

Abstract. In 1958, S. Świerczkowski proved that there cannot be a closed loop of congruent interior-disjoint regular tetrahedra that meet face-to-face. Such closed loops do exist for the other four regular polyhedra. It has been conjectured that, for any positive $\epsilon$, there is a tetrahedral loop such that its difference from a closed loop is less than $\epsilon$. We prove this conjecture by presenting a very simple pattern that can generate loops of tetrahedra in a rhomboid shape having arbitrarily small gap. Moreover, computations provide explicit examples where the error is under $10^{-100}$ or $10^{-10^5}$. The explicit examples arise from a certain Diophantine relation whose solutions can be found through continued fractions; for more complicated patterns a lattice reduction technique is needed.

1. Introduction

For each of the Platonic solids except the regular tetrahedron, it is easy to construct embedded face-to-face chains using congruent copies; embedded means that no two polyhedra have nonempty interior intersection. Figure 1 shows such toroidal loops of length 8 for octahedra, dodecahedra, and icosahedra; cubes are trivial. Steinhaus [12] wondered whether such loops exist for tetrahedra and in 1958 S. Świerczkowski [13] provided the answer: there is no such tetrahedral chain (embedded or not). We present the proof found by Dekker [5] in 1959 and, independently, Mason [10] in 1972. More details are in [15]. The main point is that the group generated by the reflections in the four tetrahedral faces is the free product $\mathbb{Z}_2*\mathbb{Z}_2*\mathbb{Z}_2*\mathbb{Z}_2$. We use $I$ for the identity matrix; if the dimension will be clear a subscript is used.

![Figure 1. Octahedral, dodecahedral, and icosahedral tori.](image)

Definition 1. A tetrahedral chain is a sequence of $k$ congruent regular tetrahedra meeting face to face, but never doubling back (i.e., $T\overline{T}\ T$ never occurs). Fixing an ordering of the four vertices of the initial tetrahedron, the chain corresponds to a sequence from $(1, 2, 3, 4)$ of length $k-1$, where the integer $i$ denotes a reflection in the $i$th face (the face opposite vertex $i$), and consecutive integers are distinct.

Theorem 1 (S. Świerczkowski). The last tetrahedron in a chain $T_0, \ldots, T_n$ cannot coincide with the first.

Proof. If $T$ is a regular tetrahedron with vertices $V_i$, let $\phi_i$ be the reflection in the face opposite $V_i$. Any point in $\mathbb{R}^3$ is uniquely representable as $x_1V_1 + x_2V_2 + x_3V_3 + x_4V_4$, where $\sum x_i = 1$; these are barycentric coordinates with respect to $T$. Each $\phi_i$ may be represented by a $4 \times 4$ matrix $M_i$ acting on these coordinates, where the columns are the vectors $\phi_i(V_j)$; $VM_i$, where $V$ is the $3 \times 4$ matrix having the $V_j$ as columns, gives the reflected tetrahedron and composition corresponds to matrix multiplication. The reflection in the face $x_i = 0$ sends $V_i$ to $C + (C - V_i) = 2C - V_i = 2\left(\frac{1}{3}\sum_{j \neq i} V_j\right) - V_i$, where $C$ is the centroid of the face opposite $V_i$ (Fig. 2), and so the barycentric matrices for the $\phi_i$ are

$$M_1 = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
2/3 & 1 & 0 & 0 \\
2/3 & 0 & 1 & 0 \\
2/3 & 0 & 0 & 1
\end{pmatrix} \quad M_2 = \begin{pmatrix}
1 & 2/3 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 2/3 & 1 & 0 \\
0 & 2/3 & 0 & 1
\end{pmatrix} \quad M_3 = \begin{pmatrix}
1 & 0 & 2/3 & 0 \\
0 & 1 & 2/3 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 2/3 & 1
\end{pmatrix} \quad M_4 = \begin{pmatrix}
1 & 0 & 0 & 2/3 \\
0 & 1 & 0 & 2/3 \\
0 & 0 & 1 & 2/3 \\
0 & 0 & 0 & -1
\end{pmatrix}$$
Let $r_0, \ldots, r_{n-1}$ be the reflection sequence of the chain. We may assume $r_0 = 1$. If $T_0$ coincided with $T_0$, then $M_1M_{r_1} \cdots M_{r_{n-1}}$ is a permutation matrix. The next claim shows that the structure of this matrix product forbids this.

Claim. Consider the product $M_1M_{r_1} \cdots M_{r_{n-1}}$ with $\frac{2}{3}$ replaced by $x$. The polynomials in the second row have $x$-degree less than $n$ except for the one in the $r_{n-1}$th column, which has degree $n$. And they all have leading coefficient $+1$.

Proof of Claim. By induction on $n$; it is clear for $n = 1$. Consider what happens when the matrix of a word that ends in $M_s$, assumed to have the claimed form, is multiplied on the right by $M_s$, with $s \neq j$. The multiplications by $x$ preserve the claimed property, as the degree becomes $s + 1$ in the $s$th position of row 2, but does not rise at all elsewhere in the row. And the leading coefficient’s sign is affected only by the $x$ multipliers.

Now look at the polynomial in the $(2, r_n)$ position: $x^n + a_1 x^{n-1} + \cdots + a_n$, where $a_i \in \mathbb{Z}$. Setting $x = \frac{2}{3}$ and taking a common denominator yields $(2^n + 3a_1 2^{n-1} + \cdots + 3^{n-1}a_n)/3^n$, the numerator of which is not divisible by 3; the fraction is therefore not 0 or 1, as required. \(\square\)

Perfection may be impossible, but searching for near-perfection is an interesting challenge. In his memoir ([14, p. 191]; see also [15, Conj. 6.1]), Świerczkowski put it this way:

“Granted then, that the last pyramid in a Steinhaus chain never can have a sideline in common with the first pyramid $P$, it still may happen that all observations and measurements indicate that these two pyramids do have a sideline in common. This would not contradict the mathematical result; it would only illustrate the obvious fact that no measurement is 100% accurate. So, a new problem is born: Whatever threshold of accuracy is selected, say, represented by a (small) number $\varepsilon$, will there be a chain of pyramids, returning to $P$ such that within the accuracy of $\varepsilon$ inches, the last pyramid of the chain has indeed a sideline in common with $P$. It is hard to tell if anyone will ever want to devote her or his time to search for an answer to this question. In any case, it is unlikely that an answer would be easily found.”

In 2015 we showed [6] how various computer searches led to chains with very small error; our best example had 540 tetrahedra with a discrepancy from closure of about $10^{-17}$ (see Fig. 17). That paper posed two challenges: Prove that the error (the discrepancy from a perfect loop) in an embedded tetrahedral chain can be made arbitrarily small; and exhibit specific examples of tetrahedral chains with error under $10^{-18}$. A reexamination of some patterns from [6] led us to the quadrilateral of §3. That chain achieves both goals, by being embedded and having arbitrarily small deviation from exact closure. And despite Świerczkowski’s prediction that this resolution will not be easy, the complete proof, once the appropriate pattern has been found, is not very complicated.

The correspondence between chains and strings from {1, 2, 3, 4} needs a little more attention in order to study the gap in a chain. For any chain $C = (T_1, T_2, \ldots, T_6)$ there is a corresponding reflection sequence $r_1, r_2, \ldots, r_{n-1}$. We wish to associate a barycentric matrix to $C$ (as in the proof of Thm. 1), so that its matrix’s distance from $I$ is related to the gap in $C$. To this end, we need an invisible starting tetrahedron $T_0$ so that we can study the matrix that takes $T_0$ to $T_6$; if that were the identity then $T_0$ would coincide with $T_6$ and $C$ would close up exactly. So consider some fixed tetrahedron $T_0$ and build the chain starting with the same legal reflection in a face of $T_0$ to get $T_1$ (see Fig. 3). There are three choices because if $r_1$ is, say, 3 then $T_0$ cannot reflect in face 3 to start the chain but it could reflect in faces 1, 2, or 4. Each of these three choices of $r_0$ yields a different reflection string $r_0, r_1, \ldots, r_{n-1}$ and a different placement of the chain in space; in particular, $T_0$ is in a different location for each choice. Each choice yields a barycentric matrix $K$ as the product $M_1M_2\ldots M_{n-1}$. To get $T_6$ from $K$, observe that its vertices are given by $T_0K$, where, abusing notation slightly, $T_0$ is the $3 \times 4$ matrix whose columns are the vertices of tetrahedron $T_0$.

We need a precise definition of the gap of a tetrahedral chain. There are many ways to do it, all starting from the idea that, for a perfectly closed loop, $T_0$ would coincide exactly with $T_6$. We can measure the difference between these two tetrahedra by looking at the Hausdorff distance $d_H$ between them as sets. For two compact sets $X$ and $Y$, $d_H(X, Y)$ is the smallest $c$ such that an expansion of $X$ by $c$ contains $Y$ and an expansion of $Y$ by $c$ contains $X$ (a $c$-expansion uses radius-$c$ balls around each point of the set). This measure can also be formulated in terms of the distance from a point to a set (see proof of Lemma 2); because our sets are tetrahedra, it is not hard, using some symbolic equation-solving on a derivative, to develop a fast algorithm to compute $d_H$ exactly for two tetrahedra.

An adequate upper bound on $d_H$ is the easily computed $d_H^{\text{corn}}$, where only the two vertex sets of the tetrahedra are used (a proof uses the alternative form of $d_H$ given in the proof of Lemma 2(a)). A chain has three associated barycentric matrices depending on the choice of initial matrix. The chain
estimate of the discrepancy from perfect closure, where the norm is the standard induced 2-norm; recall that \(||A||_2\) (often abbreviated to \(||A||\)) is sup \(||Ax||/||x||\) over nonzero vectors \(x\) and equals the largest singular value: the square root of the largest eigenvalue of \(A^TA\). Nonidentity permutations do not arise in our work, so we will omit them from the following definition. We will also use \(||K-I||_{\text{max}}\), the maximum absolute value of the matrix entries.

**Definition 2.** The gap of any chain of tetrahedra is the minimum of the three Hausdorff distances between \(T_0\) and \(T_n\), over the three choices of \(r_0\) that lead to three positions for \(T_n\). The norm gap is \(||K-I||_2\), again minimized over the three choices of \(K\).

The various notions of gap are closely related. We will use matrices so need the following bounds, which relate the gap to the norm gap and to \(||K-I||_{\text{max}}\). We need a specific invisible tetrahedron, so we will henceforth take \(T_0\) to be \(\{V_1, V_0, V_1, V_2\}\), where

\[V_i = \left(\frac{3}{10} \sqrt{3} \cos(i \theta), \frac{3}{10} \sqrt{3} \sin(i \theta), \frac{1}{\sqrt{10}} \right) \text{ with } \theta = \cos^{-1}\left(\frac{2}{3}\right)\]

as in the next section.

**Lemma 2.** Consider any tetrahedral chain with invisible tetrahedron \(T_0\). Then for any choice of the first reflection \(r_0\), with associated barycentric matrix \(K\), we have:

(a) the gap is no greater than \(||K-I||_2\);

(b) the gap is no greater than \(4||K-I||_{\text{max}}\).

**Proof.** (a) Let \(T_0\) refer to the \(3 \times 4\) matrix with the vertices of \(T_0\) as columns; let \(T_n\) be similar for the final tetrahedron, defined according to the choice of \(r_0\). The largest singular value of \(T_0\) is \(\eta = \sqrt{17}/3\), which is \(||T_0||_2\). We have \(T_0 K = T_n\). Let \(\Gamma = \{w \in [0,1]^4 : \sum w_j = 1\}\) be the set of all possible barycentric coordinates of points in a tetrahedron. Any point of a solid tetrahedron can be written as the product of that tetrahedron’s \(3 \times 4\) vertex matrix with a vector in \(\Gamma\). The Hausdorff distance is

\[d_H(T_0, T_n) = \max\{\min_{w \in \Gamma} ||T_0 u - T_0 w||, \min_{w \in \Gamma} ||T_n u - T_0 w||\};\]

the first term is bounded as follows, where all maxima and minima are over \(\Gamma\). A step at the end uses the fact that any barycentric vector lies inside the unit disk.

\[
\max_{w} \min_{u} ||T_0 u - T_0 w|| = \max_{w} \min_{u} ||T_0(K u - w)|| = \max_{w} \min_{u} ||(K - I) u|| + ||w - u||
\]

The bound on the second term is the same, proving (a).

(b) For any \(n \times n\) matrix \(A\), we have the following, where \(x\) represents a unit vector and all sums are from 1 to \(n\). We use the easily proved fact that \(\sqrt{n}\) is the maximum value of \(\sum_j |x_j|\) for a unit vector \(x\).

\[
||A||^2 = \max_{x} ||Ax||^2 = \max_{x} \left(\sum_j (a_j x_j)^2\right) \leq \max_{x} \left(\sum_j (a_j x_j)^2\right) = \left\{\max_j |a_j|^2 \right\} \sum_i \left(\sum_j (\sum_i a_{ij} x_j)^2\right) \leq \phi \max ||A||_{\text{max}} \left(\sqrt{n}\right)^2 = n^2 ||A||_{\text{max}}^2
\]

When \(n = 4\), this is ||A|| \(\leq 4 ||A||_{\text{max}}\), which, by (a), gives (b). \(\square\)

Many of our proofs rely on algebraic computation and manipulation of determinants, normal vectors, and so on. The relevant Mathematica code and some intermediate formulas are in an Appendix.

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2. The Boerdijk–Coxeter Helix

Our basic building block is the tetrahex (also called the Boerdijk–Coxeter helix), a stack of regular tetrahedra with some pleasant properties [2, 7]. Let \(r = \frac{3}{10} \sqrt{3}\), \(h = \frac{1}{\sqrt{10}}\), and \(\theta = \cos^{-1}\left(\frac{2}{3}\right) \approx 132^\circ\). The tetrahex of length \(L\) is the linear chain of \(L\) tetrahedra defined by the \(L + 3\) points \(\{V_i : i = 0, 1, \ldots, L + 2\}\), given in cylindrical form as \(V_i = (r \cos(i \theta), r \sin(i \theta), i h)\); Figure 4 shows how the vertices spiral up along the radius-\(r\) helix centered on the \(z\)-axis. Each tetrahedron has side-length 1. The reflection sequence for the tetrahex is the periodic form 1234 1234 1234 1234...
Each point $V_j$ can be represented barycentrically in terms of the basic points $T_0 = \{V_0, V_1, V_2\}$ (the invisible tetrahedron as in §1). Using the cylindrical formula, one can compute these barycentric coordinates explicitly. Then in Cartesian coordinates $V_q = T_0 C$, where $T_0$ is viewed as a $3 \times 4$ matrix with columns $V_i (-1 \leq i \leq 2)$ and $C$ is the $4 \times 1$ matrix given by

$$C = \frac{1}{10} \begin{pmatrix} 3 \\ 4 \\ 3 \\ 0 \end{pmatrix} + \frac{1}{10} \begin{pmatrix} -3 \\ -1 \\ 1 \\ 3 \end{pmatrix} q + \frac{3}{10} \begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \end{pmatrix} \cos(q \theta) + \frac{3 \sqrt{2}}{50} \begin{pmatrix} -2 \\ 1 \\ 4 \\ -3 \end{pmatrix} \sin(q \theta)$$

Note that each of the last three column vectors sums to 0 while the first sums to 1, and hence $C$ sums to 1, as is always the case for barycentric coordinates. The tetrahelix has been used in one unusual building construction; in 1989 Isozaki Arata designed a 100-meter high tower (Fig. 5) in Mito, Japan, in the shape of the tetrahelix with 28 tetrahedra [1].

![Figure 4](image)

**Figure 4.** The tetrahelix made from 16 tetrahedra colored, in order, red, green, blue, yellow, red, green, blue, yellow, and so on. The vertices are equally spaced along a helix.

**Figure 5.** A giant tetrahelix rises from the art museum in Mito, Japan.

### 3. The Quadrahelix

The *quadrahelix* $\text{QH}_L$ is built by linking four tetrahelices of length $L + 1$ using a common tetrahedron at the first and third joins and face-attachment at the second; $\text{QH}_L$ has $4L + 2$ tetrahedra. To get the appropriate reflection string, let $S_m$ denote the $m$-term tetrahelix string that begins with a 2: 2341234… 

Let $\Sigma_2$, for even $m$, be $S_{m+1}$ with its middle entry deleted; such a deletion is a way of making the appropriate first turn. Then $\text{QH}_L$ is simply $1 \Sigma_2 L \Sigma_2 L$ where $j = 3$ for even $L$ and 1 for odd $L$ and $s$ is the reversal of $s$. The corresponding barycentric matrix is the product $M_1 M_2 M_3 \cdots M_3 M_2$ defined from the full string. Thus $\text{QH}_4$ is 1 2341 3412 3 2143 1432 and $\text{QH}_{10}$ is 1 2341234123 1 2341234123 3 2143214321 3 2143214321 3 2143214321 3 2143214321 3 2143214321 3 2143214321. The deletions have the effect of making $T_{L+1}$ and $T_{3L+2}$ into pivots; they are each part of two tetrahelices. Figure 6 shows the chains for $L = 5, 6, 7, 10$; the two pivot tetrahedra are shown in gray and so each colored section has $L$ tetrahedra.
Figure 6. \(QH_4, QH_6, QH_7,\) and \(QH_{10}\). The gap for \(QH_{10}\) is 0.078, about 8% of a tetrahedron side. There are no collisions in these, and indeed none in any quadrahelix.

The basic tetrahelix is a surprising shape, looking essentially the same regardless of its length. Remarkably, the quadrahelix shares this property, in the sense that for any \(L\), the quadrahelix forms a 4-sided path having no collisions; the leg lengths (viewed along the tetrahelix axes) are all equal; and the three angles between the tetrahedral axes are \(\sec^{-1}(5), \sec^{-1}(-5),\) and \(\sec^{-1}(5)\). It exhibits several useful symmetries (see Fig. 10). The main result of this paper, that \(QH_L\) can be arbitrarily close to a closed loop, will require a proof of embeddability together with sufficient analysis to guarantee that there are values of \(L\) for which the final gap is arbitrarily small. There is in fact a single shape—the limiting rhombus—that the quadrahelices converge to, for special values of \(L\) (Fig. 13). Figure 7 shows \(QH_{70}\), a typical almost-closed chain; the gap (between red and yellow) is less than 1% of a tetrahedron side and is not visible at the full scale. For \(QH_{1990}\), the gap is \(\frac{1}{100}\%\) of a tetrahedron edge. And when \(L\) is the 99-digit integer

\[521\, 269\, 338\, 782\, 055\, 651\, 792\, 691\, 214\, 128\, 196\, 053\, 088\, 348\, 030\, 247\, 372\, 007\, 924\, 246\, 566\, 932\, 650\, 514\, 801\, 545\, 115\, 813\, 925\, 856\, 156\, 787\, 510,\]

the gap size is under \(10^{-101}\).

Figure 7. \(QH_{70}\) has a gap less than 1% of a tetrahedron side. The gap between red and yellow is shown in the magnified image at right.

Our main proof in §4 will use some exact algebraic formulas, but the underlying geometry that makes the quadrahelix work as a loop is easy to understand. Each new vertex of the tetrahelix rotates \(\theta\) around the axis. If \((L + 1)\theta\) is close to a multiple of \(2\pi\), then \(T_{L+1}\) is close to being a translation of \(T_L\); the invisible tetrahedron described in §1. Therefore the starting triangle—the floor of \(T_1\)—is almost parallel to the ceiling of \(T_{L+1}\) (Fig. 8). This in turn means that any plane orthogonal to this ceiling makes almost a right angle with the start triangle. The plane that bisects \(T_{L+1}\) as in Figure 10 (the first quadplane) then serves as a reflecting plane for the initial tetrahelix, and two more reflections cause a total change of nearly \(360^\circ\), making a nearly closed loop.
Figure 8. For certain $L$ (such as $L = 10$) the tetrahelix has a ceiling (upper plane) that is nearly parallel to the floor (plane through initial red triangle).

One can use 3D printing technology to realize these chains. Figure 9 shows a Shapeways model of $\text{QH}_{10}$, with the gap in black; this model is available at [16]. The quadrahelix pattern is so simple that it would seem not difficult to build a gap-free model of $\text{QH}_{10}$ or $\text{QH}_{29}$ or $\text{QH}_{40}$ using a standard polyhedron construction tool such as ZomeTool or Polydron.

Figure 9. A colored sandstone model of $\text{QH}_{10}$, printed by Shapeways.

4. The Disappearing Quadrahelix Gap

Our main result is that $\text{QH}_L$ is always embedded and can have an arbitrarily small gap. To prove the chains are embedded, consider $\text{QH}_L$ as falling into four congruent parts, which we think of as red, green, blue, and yellow, with red being the start of the chain. Think of the first pivot tetrahedron (gray in Fig. 10) as being half red and half green, and similarly for the pivot separating blue from yellow. The main dividing plane is the biplane, defined by the exact central triangle (the green–blue boundary in Fig. 10). The first quadplane is the one that splits the chain that lies on the side of the biplane nearest the start into two equal parts; thus it splits $T_{L+1}$ exactly in half. The second quadplane is similar, on the other side of the biplane. Thus each colored sector has $L + \frac{1}{2}$ tetrahedra, and the sectors are congruent by reflection.

Five-Plane Lemma. The biplane, first quadplane, second quadplane, start plane, and end plane all pass through a common line, called the rotation axis.
Acute Angle Lemma. Angle $\rho_0$ is acute.

The first quadplane is defined by the bisected tetrahedron atop the first tetrahelix. This means its normal vector is the edge of that tetrahedron connecting the tetrahelix vertices $P_{L+i}$ and $P_{L+i+1}$ (see Fig. 10). Thus the preceding lemma is equivalent to the next one. We define $\sigma$ to be the unique value in $[-\pi, \pi]$ that is congruent to $\sigma$ modulo $2\pi$; and we will use $\delta$ for $(L + 1)\theta$.

Acute Angle Lemma. The angle between $A$, the vector normal to the tetrahelix's base triangle $V_1V_2V_3$ and pointing in the direction of $V_1$, and the vector $V_{L+i+1} - V_{L+i}$ is acute.

Proof. A purely geometric proof can be found, but algebra is quicker. First, $A = \frac{1}{\sqrt{6}}(3V_2 - (V_0 + V_1 + V_2)) = \frac{1}{\sqrt{3}}(\sqrt{10}, 2\sqrt{2}, 3\sqrt{3})$. Using the substitution $L\theta \rightarrow \delta - \theta$ and some trig simplification yields $V_{L+i+1} - V_{L+i} = -\frac{1}{\sqrt{15}}(4\sqrt{5}\cos \delta + \sin \delta, 4\sqrt{5}\sin \delta - \cos \delta, 3\sqrt{6})$ and $A \cdot (V_{L+i+1} - V_{L+i}) = \frac{\sqrt{6}}{5}(\cos(\delta) - 1) = 2\sqrt{6}\sin^2(\frac{1}{2}\delta)$. If $\delta$ is an integer multiple of $2\pi$, then the dot product vanishes, the angle in question is exactly $90^\circ$, and the loop closes up perfectly, contradicting Theorem 1. Therefore the dot product satisfies $\frac{2}{5}\sqrt{6} \geq A \cdot (V_{L+i+1} - V_{L+i}) > 0$ and the angle is strictly between $11^\circ$ and $90^\circ$. □

Instead of Theorem 1, we could have used the irrationality of $\frac{\sqrt{5}}{2\pi}$ [10, Cor. 3.12] to deduce that $\delta$ is not a multiple of $2\pi$. In fact, the use of Theorem 1 yields an alternative proof of the irrationality result. An important corollary of the preceding proof is the following, which shows the direct connection between $\rho$ and the angle corresponding to $(L + 1)\theta$: one is small iff the other is, and the relationship is quadratic. Recall that $\delta$ abbreviates $(L + 1)\theta$, and an overbar denotes the reduced angle modulo $2\pi$.

Corollary 3(a). When $\rho_0$ is the dihedral angle between the biplane and first quadplane and $\rho = \frac{\pi}{2} - \rho_0$, we have $\cos \rho_0 = \sin \rho = \frac{2\sqrt{6}}{5}\sin^2(\frac{\delta}{2})$.

(b). If $R \in SO_3(\mathbb{R})$ is the 3-dimensional rotation matrix through angle $4\rho_0$, then $||R - I|| < \delta^2$.

Proof. (a) follows from the dot product in the acute angle lemma. (b) follows from $\cos(4\rho_0) = \cos^{-1}\left(\frac{1}{2}\sqrt{6}\sin^2(\frac{\delta}{2})\right)$ and the fact that the norm of the difference between an angle-$\alpha$ rotation matrix and the identity matrix is $\frac{1}{2}||\sin(\frac{\alpha}{2})\|$.

Theorem 4. For any $L$, the tetrahelix $QH_L$ is embedded.

Proof. A tetrahelix has no collision, so each colored sector is embedded. The next step is showing that the first sector (red) stays within the region defined by the start plane and the quad plane. This will imply that the same is true for the other colors and the appropriate planes. For this proof, we can view the end of the red sector as being the start; then the three points defining the quadplane are $V_1$, $V_2$, and $\frac{1}{2}(V_0 + V_3)$. The sign of $D$, the determinant of the $3 \times 3$ matrix $\begin{vmatrix} V_q - V_1, V_q - V_2, V_q - \frac{1}{2}(V_0 + V_3) \end{vmatrix}$, determines which side of the plane contains $V_q$. Letting $c = \cos (q \theta)$ and...
\[ s = \sin(q \theta), \text{ the matrix is this:} \]
\[
\begin{align*}
    r &\begin{pmatrix}
        c + \frac{2}{3} s - \frac{4s}{3} (q - 1) \frac{h}{r} \\
        c + \frac{1}{9} s + \frac{4s}{9} (q - 2) \frac{h}{r} \\
        c - \frac{30}{54} s - \frac{7s}{54} (q - 1) \frac{h}{r}
    \end{pmatrix}
\end{align*}
\]

and the value of \( 20 \sqrt{10} D \) works out to \( 6 \sqrt{5} q - 9 \sqrt{5} - \sqrt{5} c + 7 s \), which is not less than \( 13q - 30 \). Because \( q \geq 3 \) in this approach, \( D \) is positive, as required. The Five-Plane and Acute Angle Lemmas mean that the five planes define four regions containing the four colored sectors of the quadrahelix and there are no collisions between colors. In particular, the total angle as one moves around the rotation axis is \( 4\rho_0 < 360^\circ \) and the tetrahedra at the ends of the chain are disjoint. □

Theorem 1 shows that the gap cannot be 0. It is not hard to find nonembedded chains with arbitrarily small gaps [6, p. 61]. The fact that the quadrahe-lix is both embedded and achieves vanishingly small gaps is remarkable, considering how simple the chain is.

The next theorem concludes the proof that embedded tetrahedral chains achieve vanishingly small gaps. Our proof is algebraic, but the heart of the matter is really geometric. Figure 11 shows how the size of the jaw that, in essence, defines the gap, is close to \( 2H \sin(2\rho) \), where \( H \) is the distance (projected onto the plane perpendicular to the rotation axis) from the rotation axis to the first tetrahedron. When the jaw is small \( H \leq (4L + 2)h \) and, by Corollary 3, \( \sin \rho \leq \frac{\sqrt{L}}{10} \); these facts mean that the jaw size is bounded by \( 2L^2 \rho \), which, as shown in the proof that follows, approaches 0 for an infinite subsequence of \( L \)-values.

![Figure 11. Looking down QH_{15}’s rotation axis, which lies in five planes. Angle \( \rho_0 \) is always acute so \( \rho \) is positive.](image)

**Theorem 5.** For any \( \epsilon > 0 \), there is a quadrahelix QH\(_L\) having gap less than \( \epsilon \).

**Proof.** Recall that \( \delta = (L + 1)\theta \). If \( \delta \) were 0, then, by Corollary 3, \( \rho_0 \) would be exactly 90° and QH\(_L\) would close up perfectly: the four right angles would form a loop with zero gap. This cannot happen (by either Thm. 1 or the irrationality of \( \theta/\pi \)), but we can try to make \( \delta \) small. The convergents \( \frac{k}{q} \) of the continued fraction of \( \frac{\theta}{2\pi} \) give values \( q = L + 1 \) for which \( \frac{(L + 1)\theta}{2\pi} \) is very close to the integer \( k \). Moreover, the error \( \left| \frac{\theta}{2\pi} - \frac{k}{q} \right| \) is well known to be bounded by \( \frac{1}{\sqrt{5} q^2} \) (Hurwitz’s Thm.; [4, Exer. 7.10]). Multiplication by \( 2\pi q \) implies that \( \left| (L + 1)\theta \right| < \frac{2\pi}{\sqrt{5} (L + 1)} \). Using this method, choose \( L \) so that \( L > \frac{\epsilon}{\delta} \) and \( |\delta| < \frac{2\pi}{\sqrt{5} (L + 1)} \). An alternative to continued fractions is Kronecker’s theorem, which gives infinitely many \( L \) so that \( |\delta| < \frac{\pi}{L} \frac{2\pi}{\sqrt{5} (L + 1)} \) [8, Thm. 440].

We can use the barycentric formula (+) of §2 to get an exact symbolic expression for \( K \), the matrix giving the barycentric coordinates of the final tetrahedron of QH\(_L\). This is done in four steps. The base points are base\(_1\) = \( T_0 \) = \( \{V_{-1}, V_0, V_1, V_2\} \), where \( V_i \) are from the cylindrical formula for the tetrahelix. The base for the second leg (the tetrahedra after the first turn; green in Fig. 12) is base\(_2\) = \( \{V_{L+4}, V_{L+5}, V_{L+2}, V_{L+3}\} \). These points are used as the base in the barycentric formula and lead to the general point on the second leg, which in turn yields base\(_3\); the points on the second leg corresponding to basic tetrahelix point \( \{V_{L+3}, V_{L+2}, V_{L+1}, V_{L+4}\} \). This leads to the third leg (blue), the fourth base (which uses the same permutation as base\(_2\)), and the final leg and final tetrahedron.
Figure 12. QH_{10}, with the four barycentric base sets for the tetrahelices shown as black edges. The one between green and blue (base_3) is in the quadrabehelix; the others are not. The cyan tetrahedron is the final one in the chain.

The resulting reasonably concise formula is $K = 1 + \sigma \frac{4}{3125} (H_0 + H_1 \sigma + H_2 \sigma^2 + H_3 \sigma^3)$, where $\sigma = \sin^2 \left( \frac{\pi}{5} \right)$ and

\[
H_0 = 25 \begin{bmatrix} 6 & 21 & 6 & -9 \\ -22 & -7 & -2 & 3 \\ -2 & -7 & -2 & 3 \\ 18 & -7 & -2 & 3 \end{bmatrix} + 10 \begin{bmatrix} 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix} L + 3 \sqrt{5} \begin{bmatrix} -1 & -6 & 9 & -6 \\ 7 & 2 & -3 & 2 \\ -13 & 2 & -3 & 2 \\ 7 & 2 & -3 & 2 \end{bmatrix} \sin \delta,
\]

\[
H_1 = 2 \begin{bmatrix} -120 & -45 & 0 & 45 \\ 76 & -69 & -24 & 21 \\ 40 & 15 & 0 & -15 \\ 4 & 99 & 24 & -51 \end{bmatrix} + 600 \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} L + 3 \sqrt{5} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 6 & 26 & -14 & 6 \\ -8 & -28 & -8 & 12 \\ 2 & 2 & 22 & -18 \end{bmatrix} \sin \delta + 120 \sqrt{5} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & -2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} L \sin \delta,
\]

\[
H_2 = 48 \begin{bmatrix} -6 & -21 & -6 & 9 \\ 26 & 31 & -26 & -39 \\ -6 & 31 & -34 & 21 \\ 26 & -41 & 14 & 9 \end{bmatrix} + 10 \begin{bmatrix} -3 & -3 & -3 & -3 \\ 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & -2 & -2 \end{bmatrix} L + 3 \sqrt{5} \begin{bmatrix} 1 & 6 & -9 & 6 \\ -8 & -13 & 12 & -3 \\ 13 & 8 & 3 & -12 \\ -6 & -1 & -6 & 9 \end{bmatrix} \sin \delta.
\]

\[
H_3 = 1440 \begin{bmatrix} 4 & 3 & 0 & -3 \\ -5 & -2 & -3 & 6 \\ -2 & -5 & -6 & 3 \\ 3 & 4 & -3 & 0 \end{bmatrix}.
\]

Using this formula and assuming $L > \frac{1}{\epsilon} |\delta| < \frac{2\pi}{\sqrt{5} (L+1)}$, and also $\epsilon < \frac{1}{100}$, we next show that $4 ||K - I||_{\text{max}} \leq \epsilon$. We can bound $\sin \delta$ by $|\delta|$ and $\sin^2 \left( \frac{\pi}{2} \right) \sin \delta$ by $\frac{1}{2} \delta^2$; then the assumptions on $L$ and $\delta$ yield bounds on each of the 16 entries in $K - I$ that are polynomial in $\epsilon$. These polynomials are easily bounded in absolute value by replacing negative coefficients by positive. For example, the $(1, 1)$th entry, after replacing each symbolic coefficient (they involve rationals, $\sqrt{5}$, and powers of $\pi$) by a slightly larger approximate real is bounded by

\[
\frac{1}{4} \left( 0.95 + 3.43 \epsilon + 5.3 e^2 + 6.42 e^3 + 6.53 e^4 + 4.25 e^5 + 1.45 e^6 + 0.64 e^7 \right).
\]

The degree-7 polynomial is under 1 when $\epsilon < \frac{1}{100}$, giving the desired bound of $\frac{1}{4} \epsilon$. More detail is in the Appendix. Then Lemma 2(b) asserts that the gap of QH_\epsilon is bounded by $4 ||K - I||_{\text{max}} < \epsilon$, as required. \(\square\)

The proof yields more information. First, the barycentric formula leads to the ideal rhombus that the chains approach as $L$ increases through values so that $(L + 1) \theta \to 0$. We can choose points $P_1, P_2, P_3$ on the first three bases, do some trig expansion, replace $(L + 1) \theta$ with 0 and normalize the differences $P_3 - P_1$ and $P_3 - P_2$ to get $\frac{1}{\sqrt{(L+4)^2+1}} \begin{bmatrix} \sqrt{6} \\ -\frac{1}{3} \sqrt{6} \\ -\frac{1}{3} \sqrt{6} \end{bmatrix}$ and $\begin{bmatrix} -1 \\ \frac{1}{3} \sqrt{30} \\ \frac{1}{15} \sqrt{6} \end{bmatrix}$. The dot product of these is

$-\frac{1}{5} - \frac{19+4L}{85+40L+5L^2}$, which approaches $-\frac{1}{5}$ as $L \to \infty$. It follows that the four angles of the limiting rhombus are $\alpha^+ = \alpha^-$, $\alpha^+$, $\alpha^-$ where $\alpha^\pm = \sec^{-1}(\pm 5)$ and it is then routine to find that the normalized rhombus (Fig. 13) has vertices $(0, 0), (0, 1), \frac{1}{2} \begin{bmatrix} 2 \sqrt{5} \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \sqrt{6} \end{bmatrix}$. A consequence of this is that the limiting shape lies in a plane.
Second, we can learn some interesting asymptotic behavior as follows. Let the notation $L \to \infty$ mean that $L$ takes on only values satisfying the condition of the proof: \[ \frac{2\pi}{\sqrt{5} (L+1)} < \frac{\pi}{\delta}. \] Then \[ \lim_{L \to \infty} \frac{K-L}{\sin(\frac{\pi}{\delta})} = \frac{4}{3125} H_0, \] which is the same as \[ \lim_{L \to \infty} \frac{K-L}{\delta^2} = \frac{1}{3125} H_0. \] Also \[ \lim_{L \to \infty} \frac{h_0}{L} = 250 \begin{pmatrix} 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix} \] Therefore \[ \lim_{L \to \infty} \frac{K-L}{L \delta^2} = \frac{2}{25} \begin{pmatrix} 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix}. \] The norm of this last matrix is \( \frac{8}{25} \sqrt{3} \), so we learn that, for nearly closed quadrahelices, \( ||K - I||_2 \) is close to \( \frac{8}{25} \sqrt{3} \). Another way of looking at this is that the gap, as a fraction of the span of the whole chain, is infinitely often proportional to the square of \( (L+1)/\delta \). The chart in Figure 14, based on all values of $L$ from 4 to 200000, illustrates the convergence to \( 8 \sqrt{3} / 25 \).

![Figure 14](image)

**Figure 14.** Let $K$ be the matrix defining the last tetrahedron in QH$_2$; $\delta$ denotes $(L+1)/\delta$. The chart plots the ratio of $||K - I||_2$ to $L \delta^2$ against $\log_{10} ||K - I||_2$ for $L = 4, 5, \ldots, 200000$. Small norms are close to the product of $L \delta^2$ and \( \frac{8}{25} \sqrt{3} \).

The use of continued fractions quickly leads to nearly closed quadrahelices. Table 1 shows the convergents \( \frac{p}{q} \) to \( \frac{\pi}{2} \) and the resulting small gaps in QH$_2$, where $L = q - 1$. This method easily gives an explicit chains with $10^{10^9}$ tetrahedra and correspondingly small gap. The table shows that QH$_{20170783468,093,193}$ has smaller gap than the best chain of [6] (shown in Fig. 17).
Table 1. The left column shows the denominators of the convergents to \( \frac{\theta}{2\pi} \), less 1; these yield almost closed chains \( \text{QH}_n \). The second column has the numerators, the multiples of \( 2\pi \) that reduce \( (L+1)\theta \) to near 0. The third column shows the reduced angle \( (L+1)\theta \) and the last gives the gap size, measured by Hausdorff distance.

5. Algebra and Geometry

An analysis of the matrix \( K \) ties together several algebraic and geometric facts about a certain family of tetrahedral chains (which includes the quadrahelix and octahelix). Suppose \( C \) is a chain of \( n \) tetrahedra \( T_i \) that is symmetric by reflection in a plane \( \Pi \) through one of its faces; then \( n \) is even and the reflection face is the middle face. Let \( F \) denote reflection in \( \Pi \); \( F \) takes \( T_1 \) to \( T_n \). Let \( F' \) be the reflection in the face \( \Pi' \) of \( T_1 \) that gives the invisible tetrahedron \( T_0 \). By Theorem 1, these two planes are not parallel. Let \( \Omega \) be the isometry taking \( T_0 \) to \( T_n \). Then \( \Omega = F F' \) and therefore \( \Omega \) is a rotation. The axis of \( \Omega \) is the line \( \Pi \cap \Pi' \) (Proof: for \( p \) on this line, \( \Omega(p) = F F'(p) = p \)). This explains why our constructions have the useful rotation axis defined by certain planes (see Fig. 10). The chain is embedded if its first half is embedded and lives on one side of \( \Pi \).

Although \( T_n \) was defined to be the vertex set of a tetrahedron in a chain, we will also use it for the \( 3 \times 4 \) matrix with these vertices as columns. With \( K \) as the barycentric matrix of the chain (the product of the \( M_i \)), recall that the final tetrahedron \( T_0 \) is given by \( T_n = T_0 K \). This transformation of the invisible tetrahedron to the final tetrahedron can be viewed as a transformation of the \( 4 \times 4 \) matrix \( T_0 = \begin{bmatrix} T_n & T_0 \end{bmatrix} \) to \( T_n = \begin{bmatrix} T_0 & T_0^{-1} \end{bmatrix} \). One way to effect this transformation is by \( T_0 K = T_n \). Another way is to consider the decomposition of \( \Omega \), the rotation induced by the motion; \( \Omega \) decomposes into \( \tau \cdot R \) where \( R \) is a rotation that fixes the origin and \( \tau \) is a translation by \( i \). Therefore, letting \( R = \begin{bmatrix} R & \begin{bmatrix} \\[1\end{bmatrix} 
\end{bmatrix} \), we have \( \mathcal{R} T_0 = T_n \). The two approaches combine to give \( \mathcal{R} T_0 = T_0 K \) and \( \mathcal{R} = T_0 K T_0^{-1} \). For \( \text{QH}_n \), we derived a symbolic expression for \( K \); for chains in general \( K \) is easy to compute as a product. So the preceding relation gives both a symbolic expression and a simple method of computation for \( \mathcal{R} \) and \( i \).

The eigenvalues of \( \mathcal{R} \) (and hence also \( K \)) are \( z, z, 1, 1 \). The two left fixed points of \( \mathcal{R} \) are \((0, 0, 0, 1) \) and \((w_1, w_2, w_3, 0) \) where, letting \( w = (w_1, w_2, w_3), w^T R = w^T \) and \( w^T i = 0 \). The corresponding two fixed points of \( K \) are \((0, 0, 1) T_0 = (1, 1, 1, 1) \) and \((w_1, w_2, w_3, 0) T_0 = w^T T_0 \). One can get \( w \) as the normalized cross product of the largest (in norm) columns of \( R - I \); so \( w \) is a unit vector.

We learn from the preceding that:

- \( R \) and \( i \) are easily computed from \( K \).
- \( \pi \) gives the direction of the axis of \( R \) and \( \Omega \) and is easily computed as an eigenvector of \( R \).
- The rank of \( K - I \) is 2 (it is not 1 because its eigenvalues are \((z, z, 1, 1) \) = \((z - 1, z - 1, 0, 0) \). The left kernel of \( K - I \) is generated by \((1, 1, 1, 1) \) and \( w^T T_0 \).

To fully characterize the isometry \( \Omega \) we need a point on its axis; such a point \( t \) satisfies \( t - R t = i \). But \( R - I \) has rank 2 so is singular (as is true of any \( R \in SO_3(R) \)), so we cannot get \( t \) by inversion. But fairly standard linear algebra gives \( t \) as follows. Let \( i^* \) be \( i \) normalized to have length 1. Then \( t^* = (1 - 2 \pi i^*)^{-1} Q_i \), where \( Q_i \) is the \( 3 \times 2 \) matrix whose first column is \( i^* \) and second column is \( w^T i^* \). So now \( \Omega \) is completely characterized: \( \Omega \) is a rotation around the axis in direction \( \pi \) through the point \( t^* \) and by an angle that is \( \arg(v) \) where \( v \) is an eigenvalue of \( R \); thus the line \( \Pi \cap \Pi' \) is given by \( \pi + \alpha \pi \).

These general ideas can be applied to the quadrahelix and lead to a different proof of Theorem 5, one that has much in common with the geometric proof mentioned prior to that theorem.

- By Corollary 3, \( ||R - I|| \leq \delta^2 \), where \( \delta \) is the mod-\( 2\pi \) reduction of \( (L+1) \theta \).
• The proof of the acute angle lemma showed that \( \rho_0 \approx \cos^{-1}\left(\frac{\sqrt{6}}{3}\right) \) applying the Law of Cosines to an isosceles triangle with apex angle \( 2\rho_0 \) and opposite side bounded by \((2L + 1)h\) yields

\[
\|\sigma\| \leq \frac{h(2L + 1)}{\sqrt{2}} \sqrt{1 - \cos^2\left(\frac{2\pi}{\theta}\right)} = \frac{h(2L + 1)}{\sqrt{2}} \sqrt{1 - \frac{2}{\sqrt{3}}^2} = \frac{5}{2} h (2L + 1) \leq 0.79 + 1.59 L
\]

If we assume \( L \geq 17 \), then \( 1 + \|\sigma\| \leq 1.7L \).

• \( \|T_0\| = \frac{1}{\sqrt{2}} \sqrt{117 + \sqrt{8689}} \approx 2.06 \)

• gap \( \leq \|K - I\| \leq \frac{1}{\sigma_{\min}(T_0)} \|T_0 (K - I)\| = \sqrt{2} \|T_0 - T_0\| \), where \( \sigma_{\min} \) denotes the smallest singular value.

• Let \( R_{3,4} = [(R - I) \cdot \tilde{t}] \), where \( \cdot \tilde{t} \) indicates the adjunction of the column \( \tilde{t} \); then \( T_0 - T_0 = R_{3,4} T_0 \).

These facts and an application of the induced 2-norm to the last equality give the following, assuming \( L \geq 17 \).

\[
\frac{1}{\sqrt{2}} \text{gap(OH}_L) \leq \|T_0 - T_0\| = \|R_{3,4} T_0\| \leq \|[(R-I) - (R-I)\sigma]\| \|T_0\| \\
\leq \|R - I\| \left(1 + \|\sigma\|\right) \|T_0\| \\
\leq 1.7 \left(1 + \|\sigma\|\right) \approx 3.51 \leq 3.51 L \delta^2
\]

So this bounds the gap by \( 3.51 L \delta^2 \). Because \( \delta \) is less than \( 1/L \) infinitely often (see proof of Thm. 5) this provides an alternate proof that the gap of QH\(_L\) can be made arbitrarily small.

6. An Octagonal Pattern

We first resolved the vanishing-gap conjecture with an 8-sided shape that is more complicated than the quadrahelix. The octahelix OH\(_L\) arises from the 8-part sequence \( S_{l+1} S_l p(S_{l+1}) p(S_l) S_{l+1} S_l p(S_{l+1}) p(S_l) \), where \( S_\alpha \) is the \( m \)-term string 12341234… and \( p \) is the permutation \([2, 3, 4, 1]\), a 4-cycle. Thus OH\(_4\) is 1234 4321 2341 2341 1432 1234 4321 2341 1432. The useful symmetries of the shape are clarified if we shift the string left \( L \) units; that is, use \( j \) \( S_L p(S_{l+1}) S_l p(S_{l+1}) S_l p(S_{l+1}) S_l p(S_{l+1}) S_l \) for OH\(_L\), where \( j \equiv 1 + L \) (mod 4). The cases \( L = 4, 5, 6, \) and 36 are shown in Figure 15.

![Figure 15](image-url) The octahelix for \( L = 4, 5, 6, \) and 36; OH\(_L\) is not embedded. The gaps for OH\(_6\) and OH\(_{36}\) are both about 0.02.

The analysis is more complicated than for the quadrahelix, but the general approach is similar; there are five important planes (really nine, since the planes bisecting each colored segment are useful too), an acute angle lemma, a five-plane lemma, an embedding theorem (which fails for \( L = 1 \) or 4), two magic angles, and a number-theoretic relation that leads to small gaps. There are two magic angles, \( \gamma = \cos^{-1}\left(\frac{1}{12}(-3 \pm 5 \sqrt{3})\right) \); when \( L \theta \) is near either one, the gap is small. If \( \delta = L \theta - \gamma \) and assuming \( |\delta| < 1 \), one can show that the gap is bounded by \( 3L \delta^2 \). For example, if \( L = 686 \), then \( 686 \delta = 0.000035... \), the bound is 0.0003, and the actual gap in OH\(_{686}\) is about 0.000016. A pleasant property of the octahelix is that all seven angles equal \( \sec^{-1}(5) \).

The octahelix construction is quite similar to the quadrahelix, except that for QH there was only one magic angle, \( -\theta \), for that case. When \( L \theta \) is close to \( -\theta \), the quadrahelical gap is small. Having \( \theta \) be both the magic angle and the multiplying angle is wonderful because \( L \theta + 2\pi k \sim -\theta \) becomes \( (L + 1) \theta - 2\pi k \) and the continued fraction of \( \frac{\theta}{2\pi} \) gives nearly closed quadrahelices. For the octahelix, we need the more complicated relation \( L \theta + 2\pi k \sim \gamma \). Kronecker’s theorem and the density in the unit circle of the set \( [L\theta] \) can be used as mentioned in §4 and yields a proof that for a subsequence of \( L\)-values, OH\(_L\) is embedded and has vanishingly small gap. But to get explicit \( L\)-values requires more sophistication than the continued fraction method. Danny Lichtblau [9] developed a method that uses lattice reduction to solve this problem. We present the details because...
his algorithm is useful and interesting. As with the quadrahelix, this method can find explicit values of \( L \) for which the gap in \( \text{OH}_L \) is \( 10^{-100} \) or smaller.

**Lichtblau’s method for the Diophantine relation** \( x \alpha + y \beta + \gamma \sim 0 \)

We have \( \theta = \cos^{-1}\left(-\frac{1}{3}\right) \) and \( \gamma^* = \cos^{-1}\left(\frac{1}{12}\left(5 + \sqrt{3}\right)\right) \) and seek \( L \) so that \( L \theta \) is close to \( \gamma^* \). This is the same as \( L \theta + k(2\pi) \) being close to \( \gamma^* \), which is a special case of the more general problem: Given \( \alpha, \beta, \gamma \in \mathbb{R} \) with irrational \( \frac{\alpha}{\beta} \), find \( x, y, z \in \mathbb{Z} \) so that \( x \alpha + y \beta + z \gamma \) is close to \( 0 \). Such integers do exist: Kronecker’s Theorem [8, Thm. 440] gives \( n, m \in \mathbb{Z} \) so that \( |n \frac{\alpha}{\beta} + m + \frac{\gamma}{n}| < \frac{3}{n} \), or \( |n \alpha + m \beta + y \gamma| < \frac{3}{n} \). In our problem \( \alpha = \theta, \beta = 2\pi, \) and \( \gamma = -\gamma^* \). Table 2 shows the values of \( n \) under \( 10^9 \) that satisfy the estimate. Because \( \frac{\alpha}{\beta} \) is close to \( \frac{\gamma}{2\pi} \), or about 0.36, \( n, m \), and the reciprocal of the error will have the same order of magnitude; this can be seen in the last column of the table.

![Table 2](image)

**Table 2. Solutions to** \( |n \theta + m 2 \pi - \gamma^*| < \frac{6 \pi}{n} \). The values of \( n, m \), and the error reciprocal have the same order of magnitude.

Given a Kronecker theorem example, define a scaling factor \( s \) to be the reciprocal of the error bound \( \frac{3\theta}{s} \); so \( n = \frac{3\theta}{s} \). Use \( s \) to get integers \( a = [s \alpha], b = [s \beta], c = [s \gamma] \) and let \( \delta_a, \delta_b, \delta_c \) be the differences \( a - s \alpha \), and so on; so \( 0 \leq \delta_a < 1 \) and \( a = \frac{1}{s}(a - \delta_a) \), etc. This leads to the Diophantine equation \( u a + x b + y c = 0 \), where \( u \) is introduced to quantify the error in the approximation. By Kronecker, we have the following:

\[
|n \alpha + m \beta + y \gamma| < \frac{3 \beta}{n}
\]

\[
\left|n \frac{1}{s} (a - \delta_a) + m \frac{1}{s} (b - \delta_b) + \frac{1}{s} (c - \delta_c)\right| < \frac{1}{s}
\]

\[
\left|n (a - \delta_a) + m (b - \delta_b) + (c - \delta_c)\right| = \left|(n a + m b + c) - (n \delta_a + m \delta_b + \delta_c)\right| < 1
\]

This means that the integer \( n a + m b + c \) is the ceiling of \( n \delta_a + m \delta_b + \delta_c \). But the magnitude of this real is bounded by \( n + m + 1 \). Therefore \( n a + m b + c \) is about \( n + m \). So there exists a solution \((u, x, y, z)\) to the Diophantine equation \( u a + x b + y c = 0 \) where \( u, x, y \), and the scaling factor have the same order of magnitude.

Hermite decomposition is a classic way to solve this Diophantine problem, but the magnitude condition will typically fail. In our example, consider \( s = 10^{10} \); then \([a, b, c] = [23005239830, 62831853072, -1079592371111] \). The Hermite method gives \((u, x, y, z) = (-1, 20188605320, -739185179, 1)\). The error component \((u = -1)\) is very small, but 20188605320 is much larger than \( s \) and so these integers do not translate to a desired solution to the irrational approximation problem. The ceiling-error gets magnified too much in the move to the reals.

Lattice reduction can find the solution we want. Let \( A = \begin{pmatrix} -23005239831 & 1 & 0 & 0 \\ -62831853072 & 0 & 1 & 0 \\ -10795923711 & 0 & 0 & 1 \end{pmatrix} \) a lattice generated by the rows. Each row of \( A \) is a solution to the equation. A minimal basis for a lattice \( \Lambda \) has vectors \( b_i \) so that \( ||b_i|| \) is minimal among all vectors in \( \Lambda \). If \( ||b_i|| \) is minimal among all vectors that are not multiples of \( b_i \), and so on. Finding the minimal basis is difficult, but the famous lattice reduction algorithm (Lovász–Lenstra–Lenstra; see [3]) can quickly find a reduced basis, one that is close to minimal.

The reduced basis of \( A \) is \( \begin{pmatrix} 1179 & -2322 & 661 & -1101 \\ -2673 & -847 & -216 & -3062 \\ 3978 & 3229 & -1590 & -2373 \end{pmatrix} \), giving three solutions. The first row gives \( 2322a - 661b + 1101c = 1179 \), the
smallest error, but $z$ is 1101, not 1, which is not helpful because we require that $z$ (the multiplier of $c$) be 1. So we take the initial lattice instead to be

$$A = \begin{pmatrix} -23005 & 239831 & 1 & 0 & 0 \\ -62831 & 853072 & 0 & 1 & 0 \\ 10795923711 & 0 & 0 & w \end{pmatrix}$$

for some integer weight $w$. Now each row of $A$ solves $u + xa + yb + z \left( \frac{1}{w} \right) = 0$ and therefore the same is true of the reduced basis, whose vectors are all short. But we need to have $z = w$ so that the multiplier of $c$ is 1. Recall that Kronecker’s Theorem tells us that a solution does exist.

When $w = 2$, we get

$$\begin{pmatrix} 1179 & -2322 & 661 & -2202 \\ -3852 & 1475 & -877 & -3922 \\ 2799 & 5551 & -2251 & -2544 \end{pmatrix}$$

The best row has $z = -2202$, which differs from $w$ by a factor of 1101, no reduction from the previous case. But when $w = 3$ the reduced basis is

$$\begin{pmatrix} 1179 & -2322 & 661 & -3303 \\ 2799 & 5551 & -2251 & -3816 \\ -5031 & 3797 & -1538 & -2580 \end{pmatrix}$$

and $2580/w = 860$ for a small improvement.

Computations (Fig. 16) show that the quotients $\frac{z}{w}$ only decrease as the weight rises. At $w = 29645$ the reduced basis is

$$\begin{pmatrix} -45972 & 14570 & -5335 & -59290 \\ -78813 & 64708 & -23692 & \text{29,645} \\ 150039 & 184557 & -67574 & -59290 \end{pmatrix}$$

and the goal is achieved: the second row ends with $w$, the $c$-coefficient is 1, and the key value is 64708, which appears as a good value in Table 2. Figure 16 shows a log-log plot of the ratio $\frac{z}{w}$; Table 3 shows the good $L$-values that arise from this method. We used $w = \left\lfloor \sqrt{5} \right\rfloor$ for the weight, as that appears to do the job. Again, the reason all this works is that, by Kronecker, a solution of the desired form does exist; therefore it is not surprising that lattice reduction finds it.

![Figure 16](image.png)

Figure 16. The plot shows how, as the weight increases, the reduced basis converges to one with a vector whose last entry equals the weight; here this occurs first at $w = 29645$.

One can even get by this method a million-digit value of $L$ so that the gap in $O_{L}$ is under $10^{-10^{9}}$. When lattice reduction gives a vector with negative $L$, it is easy to see that using the positive value $-L - 1$ leads to $L\beta$ being close to $\gamma^\circ$ (as opposed to $\gamma^+$), and this too yields a small gap.

<table>
<thead>
<tr>
<th>scale factor $s$</th>
<th>$L$</th>
<th>$k$</th>
<th>$L+2\sqrt{c-L}$</th>
<th>upper bound on $\log_{10}$ gap</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{6}$</td>
<td>-5</td>
<td>2</td>
<td>-0.032</td>
<td>-2</td>
</tr>
<tr>
<td>$10^{8}$</td>
<td>-35</td>
<td>13</td>
<td>0.083</td>
<td>0</td>
</tr>
<tr>
<td>$10^{10}$</td>
<td>432</td>
<td>-158</td>
<td>0.0035</td>
<td>-2</td>
</tr>
<tr>
<td>$10^{12}$</td>
<td>686</td>
<td>-251</td>
<td>0.00035</td>
<td>-4</td>
</tr>
<tr>
<td>$10^{14}$</td>
<td>38707</td>
<td>-14172</td>
<td>0.000045</td>
<td>-4</td>
</tr>
<tr>
<td>$10^{16}$</td>
<td>64708</td>
<td>-23692</td>
<td>3.3 x $10^{-8}$</td>
<td>-6</td>
</tr>
<tr>
<td>$10^{18}$</td>
<td>-4292893</td>
<td>1571799</td>
<td>5.2 x $10^{-7}$</td>
<td>-5</td>
</tr>
<tr>
<td>$10^{20}$</td>
<td>19758906</td>
<td>-7234521</td>
<td>4.6 x $10^{-8}$</td>
<td>-7</td>
</tr>
<tr>
<td>$10^{22}$</td>
<td>-65209617</td>
<td>238733970</td>
<td>-1.2 x $10^{-8}$</td>
<td>-7</td>
</tr>
<tr>
<td>$10^{24}$</td>
<td>-3113400371</td>
<td>1139939676</td>
<td>-2.0 x $10^{-10}$</td>
<td>-9</td>
</tr>
<tr>
<td>$10^{26}$</td>
<td>58842821865</td>
<td>-21544696887</td>
<td>1.7 x $10^{-18}$</td>
<td>-8</td>
</tr>
<tr>
<td>$10^{28}$</td>
<td>53229276314</td>
<td>-194893545793</td>
<td>1.8 x $10^{-12}$</td>
<td>-11</td>
</tr>
<tr>
<td>$10^{30}$</td>
<td>2745915283354</td>
<td>-1005388772711</td>
<td>-4.9 x $10^{-13}$</td>
<td>-12</td>
</tr>
</tbody>
</table>

Table 3. Lattice reduction leads to values of $L$ for which the octahelix has a small gap.

7. Open Questions

A natural question is whether our quadrahelix can be improved. There are several aspects to consider.

**Question 1.** Is there a triangular pattern that leads to embedded chains with vanishing gaps?

Some investigations indicate that the quadrahelix idea will not work for a pattern based on an equilateral triangle or a square. But there might be something involving other types of triangles, or a rectangle, or perhaps a completely sporadic sequence of triangles that works.

The quadrahelix requires roughly $6/\epsilon$ tetrahedra to achieve a gap of $\epsilon$; it takes about $10^{17}$ tetrahedra to get a quadrahelical gap near $10^{-17}$. In [6] we found an embedded chain of 540 tetrahedra with a gap of $7 \cdot 10^{-18}$ (see Fig. 17). The reflection sequence for this chain is
(2 · 22 + 1) · 4 · 3 = 540. Experiments [6, Fig. 9] suggest that a gap of $\epsilon$ can be achieved with $10 \log(1/\epsilon)$ tetrahedra, a formula that predicts about 400 tetrahedra for $\epsilon = 10^{-17}$.

**Question 2.** Is there a pattern for embedded chains that achieves gap $\epsilon$ using $O(\log(1/\epsilon))$ tetrahedra?

![Figure 17. An embedded chain of 540 tetrahedra that has a gap smaller than 10$^{-17}$.](https://www.researchgate.net/publication/228393149_Solving_knapsack_and_related_problems)

A main point about our symmetric patterns is that the gap depends on two things: how close the rotation angle induced by the chain is to $2\pi$, and how large $L$ is. More generally, it would seem that the role of $L$ might be replaced by the overall diameter of the chain, so that one line of attack on Question 2 is to look for chains that use a large number of tetrahedra but have them arranged much more compactly than the quadrahelix or octahelix.

Finally, we ask whether the lattice reduction approach can be designed so that it provably solves any Diophantine problem of the sort that arose in the study of the octahelix.

**Question 3.** Can one prove that, given irrational $\alpha, \beta$, the lattice reduction method of §6 can always be used with some scaling factor and weight to get integers so that $|x \alpha + z \beta + y|$ is small?

**References**

Appendix: Formulas and Mathematica Code

Basic Utilities

Clear[a, b, c, x1, y1, z1, x2, y2, z2, x3, y3, z3, x4, y4, z4];
bary[{{x1_, y1_, z1_}, {x2_, y2_, z2_}, {x3_, y3_, z3_}, {x4_, y4_, z4_}}] :=
Evaluate[With[{pts4 = {{x1, y1, z1}, {x2, y2, z2}, {x3, y3, z3}, {x4, y4, z4}}, pt = {a, b, c}, var = x /@ Range[4]},
Simplify[var / Solve[{Total[var] = 1, And @@ Thread[var.pts4 = pt]}, var] // l]]];
fromBary[bary_, pts4_] := bary.pts4;
θ = ArcCos[-2/3]; r = 3 Sqrt[3]; h = 1/10; id = IdentityMatrix[4];
V[i_] := (r Cos[i θ], r Sin[i θ], i h);
base1 = V /@ (-1, 0, 1, 2);

The Barycentric Formula for the Tetrahelix

CC = Simplify[bary[base1, V[Q]]] /. (Cos[θ Q] → c, Sin[θ Q] → s);
(coeffQ, coeffc, coeffs) = Coefficient[CC, θ] & /@ (Q, c, s);
con = Expand[CC - (coeffQ Q + coeffc + coeffs s)]

\[
\begin{bmatrix}
-3 & -1 & -1 & 3 \\
10 & 10 & 10 & 10 \\
-1 & -3 & 3 & 3 \\
10 & 10 & 10 & 10 \\
\end{bmatrix},
\begin{bmatrix}
-3 & -3 & 3 & 3 \\
3 & 3 & 3 & 3 \\
5 & 5 & 5 & 5 \\
10 & 10 & 10 & 10 \\
\end{bmatrix},
\begin{bmatrix}
-3 & 3 & 3 & 3 \\
5 & 5 & 5 & 5 \\
10 & 10 & 10 & 10 \\
\end{bmatrix}
\]

The Dot Product in the Acute Angle Lemma

A = TrigExpand[3 V[3] - Total[{V[0], V[1], V[2]}]]

\[
3 \sqrt{5} A
\]

\[
\begin{bmatrix}
2 & 2 & 2 \\
3 & 3 & 3 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 \\
10 & 2 \sqrt{2} & 3 \sqrt{3} \\
\end{bmatrix}
\]


Simplify[\[SquareRoot]15 3 B] /. θ L → δ

\[
\begin{bmatrix}
\frac{1}{45} \left(20 \sqrt{3} \cos \left[\text{ArcCos}\left(\frac{2}{3}\right)\right] + \sqrt{15} \sin \left[\text{ArcCos}\left(\frac{2}{3}\right)\right]\right), \\
\frac{1}{45} \left(-\sqrt{15} \cos \left[\text{ArcCos}\left(\frac{2}{3}\right)\right] - 20 \sqrt{3} \sin \left[\text{ArcCos}\left(\frac{2}{3}\right)\right]\right), \\
\frac{2}{5}
\end{bmatrix}
\]

\[
\begin{bmatrix}
4 \sqrt{5} \cos[δ] + \sin[δ], -\cos[δ] + 4 \sqrt{5} \sin[δ], 3 \sqrt{6}
\end{bmatrix}
\]

Factor[TrigExpand[A.B /. L θ → δ - θ]]

TrigFactor[TrigExpand[A.B /. L θ → δ - θ]]

\[
\begin{bmatrix}
\frac{1}{5} - \sqrt{6} \left(-1 + \cos[δ]\right) \\
\frac{2}{5} \sqrt{6} \sin[δ]^{2}
\end{bmatrix}
\]
The Determinant for the Embedding Theorem

\[
\text{mat} = \text{TrigExpand}\left[\{\text{V}[Q] - \pi \& \rightarrow \{\text{V}[1], \text{V}[2], \frac{1}{2} (\text{V}[0] + \text{V}[3])\}\right];
\]
\[
\text{MatrixForm[mat /. \(\theta \rightarrow \theta\)]}
\]
\[
\text{X} = \text{Expand}\left[20 \sqrt{10} \text{Det[mat]}\right];
\]
\[
\text{X} /. \(\theta \rightarrow \theta\)
\]
\[
\left\{\begin{array}{l}
\frac{\sqrt{5}}{5} + \frac{3}{10} \sqrt{3} \cos(\theta Q) - \frac{\sqrt{7}}{2} + \frac{3}{10} \sqrt{3} \sin(\theta Q) - \frac{1}{\sqrt{10}} + \frac{1}{\sqrt{10}} \\
\frac{1}{10} + \frac{3}{10} \sqrt{3} \cos(\theta Q) + \frac{1}{10} - \frac{2}{12} \sqrt{5} \sin(\theta Q) - \frac{1}{\sqrt{10}} + \frac{1}{\sqrt{10}} \\
-9 \sqrt{5} + 6 \sqrt{5} Q - \sqrt{5} \cos(\theta Q) + 7 \sin(\theta Q)
\end{array}\right.
\]
\[
N[\text{X} /. \(s \rightarrow (\cos[\sin][\_]) \rightarrow -\text{Abs}[\_]\)]
\]
\[
-29.3607 \rightarrow 13.4164 Q
\]

The Symbolic Matrix \(K\) of the Quadrahelix

First get the barycentric form of a general point \(V_Q\) on the tetrahelix. It is not too complicated.

\[
\text{baryGen} = \text{Simplify[TrigExpand[ExpandAll[fromBary[baryGen, base2]]]], (Q, L \rightarrow (0, 3, 2, 1))};
\]
\[
\text{fina} = \text{Transpose[finaTetBaryForm]};
\]

Next is therefore an exact symbolic form of the barycentric matrix \(K\) for \(QH\), where \(s = \sin \delta, \sigma = \sin^2(\frac{\delta}{2})\) and \(\delta = (L + 1) \theta.\)
\[
\begin{array}{l}
\left\{ \begin{array}{l}
3125 \cdot 300 \left( 15-10 \sqrt{5} \right) \sigma =-24000 \sigma^2-578 \left( 10-15 \sqrt{5} \right) \sigma=-23040 \sigma^2
\end{array} \right.
\end{array}
\]
\[
\frac{12 \sigma \left( -25-48 \sigma \right)}{625 \sqrt{5}} \cdot \left( 2 \sqrt{5} L-6 \sigma+\sqrt{5}(-7-4 \sigma) \right) = \frac{12 \sigma \left( -25-48 \sigma \right)}{625 \sqrt{5}} \cdot \left( 2 \sqrt{5} L-6 \sigma+3 \sqrt{5}(-1-2 \sigma) \right)
\]
\[
\frac{1}{625 \sqrt{5}} \cdot 4 \sigma \left[ 3 \left( -175 - 120 \sigma + 384 \sigma^2 \right) + L \left( 50 \sqrt{5} + 240 \left( \sqrt{5} - \sigma \right) \sigma - 384 \sqrt{5} \sigma^2 \right) + 2 \sqrt{5}^2 \left( 275 - 380 \sigma - 624 \sigma^2 + 720 \sigma^3 \right) \right]
\]
\[
\frac{1}{625 \sqrt{5}} \cdot 4 \sigma \left[ 625 \sqrt{5} - 100 \left( 7 \sqrt{5} + 2 \sqrt{5} L - 6 \sigma \right) \sigma + 120 \left( 23 \sqrt{5} + 8 L \left( \sqrt{5} - \sigma \right) \right) \sigma^2 + 192 \left( 31 \sqrt{5} + 8 \sqrt{5} L - 39 \sigma \right) \sigma^3 - 2304 \sqrt{5} \sigma^4 \right]
\]
\[
\frac{1}{625 \sqrt{5}} \cdot 4 \sigma \left[ 6 \left( 225 + 840 \sigma - 1728 \sigma^2 \right) + L \left( 50 \sqrt{5} + 240 \left( \sqrt{5} - \sigma \right) \sigma - 384 \sqrt{5} \sigma^2 \right) + 2 \sqrt{5} \left( 25 - 120 \sigma - 624 \sigma^2 + 432 \sigma^3 \right) \right]
\]
\[
\frac{1}{625 \sqrt{5}} \cdot 4 \sigma \left[ 6 \left[ 50 \sqrt{5} - 240 \left( \sqrt{5} - \sigma \right) \sigma - 384 \sqrt{5} \sigma^2 \right] + 3 \left[ 50 \cdot 120 - 144 \sigma^2 \right] - 2 \sqrt{5} \left[ 25 - 70 \cdot 624 \sqrt{5}^3 \sigma^6 \right] \right]
\]
\[
\frac{1}{625 \sqrt{5}} \cdot 4 \sigma \left[ 6 \left( 975 - 480 \sigma - 1872 \sigma^2 \right) + 2 \sqrt{5} \left( 25 - 200 \sigma + 144 \sigma^2 - 288 \sigma^3 \right) - L \left( 480 \sigma - 2 \sqrt{5} \left( 25 - 48 \sigma \right) \right) \right]
\]
\[
\frac{1}{625 \sqrt{5}} \cdot 4 \sigma \left[ 6 \left( 25 - 200 \sigma + 192 \sigma^2 \right) + \sqrt{5} \left[ 175 - 150 \sigma + 144 \sigma^2 + 144 \sigma^3 \right] - L \left[ 480 \sigma - 2 \sqrt{5} (25 - 48 \sigma) \right] \right]
\]
\[
\frac{1}{625 \sqrt{5}} \cdot 4 \sigma \left[ 6 \left( 25 - 200 \sigma + 192 \sigma^2 \right) + \sqrt{5} \left( 34 \sqrt{5} - 25 \sqrt{5} L - 9 \sigma \right) \sigma^4 - 6912 \sqrt{5} \sigma^6 \right]
\]
\[
\frac{1}{625 \sqrt{5}} \cdot 4 \sigma \left[ 6 \left( 480 \sigma - 2 \sqrt{5} \left( 25 - 48 \sigma \right) \right) - 1 \left[ 50 \cdot 240 - 576 \sigma^2 \right] + \sqrt{5} \left[ 25 - 50 \cdot 336 \sigma^2 - 280 \sigma^3 \right] \right]
\]
\[
\frac{1}{625 \sqrt{5}} \cdot 4 \sigma \left[ 6 \left( 525 - 120 \sigma - 864 \sigma^2 \right) - L \left[ 50 \sqrt{5} - 240 \left( \sqrt{5} - \sigma \right) \sigma - 384 \sqrt{5} \sigma^2 \right] + 2 \sqrt{5} \left[ 225 - 20 \cdot 624 \sqrt{5}^3 \sigma^6 \right] \right]
\]
\[
\frac{1}{625 \sqrt{5}} \cdot 4 \sigma \left[ 6 \left( -6 \sigma + 25 - 20 \sigma + 24 \sigma^2 \right) + L \left[ -50 \sqrt{5} + 240 \left( \sqrt{5} - \sigma \right) \sigma - 192 \sqrt{5} \sigma^2 \right] + \sqrt{5} \left[ -175 - 990 \sigma - 1968 \sigma^2 + 1152 \sigma^3 \right] \right]
\]
\[
\frac{1}{625 \sqrt{5}} \cdot 4 \sigma \left( 3 \left( 75 - 440 \sigma + 288 \sigma^2 \right) + 2 \left( 25 \sqrt{5} - 120 \left( \sqrt{5} - \sigma \right) \sigma + 96 \sqrt{5} \sigma^2 \right) + 2 \sqrt{5} \left( 25 - 120 \sigma - 336 \sigma^2 + 432 \sigma^3 \right) \right]
\]
\[
\frac{1}{625 \sqrt{5}} \cdot 4 \sigma \left( 625 \sqrt{5} - 100 \left( 2 \sqrt{5} L - 3 \left( \sqrt{5} + 2 \sigma \right) \sigma - 120 \left( -17 \sqrt{5} - 36 \sigma + 8 L \left( \sqrt{5} + \sigma \right) \sigma^2 - 192 \left( 4 \sqrt{5} L - 9 \left( \sqrt{5} + 3 \sigma \right) \sigma^2 \right) \right) \right) \right)
\]

A Compact Form of K

```
raw1 = Expand[raw / σ];
cs = CoefficientList[raw1, σ];
Do[cl[k] = Table[Coefficient[cs[i, j, k + 1] L, {i, 4}, {j, 4}];
  c[k] = Table[Coefficient[cs[i, j, k + 1] s, {i, 4}, {j, 4}];
  cCon[k] = Simplify[Table[c[l][i, j, k + 1] - cl[k][i, j] L - cs[k][i, j] s, {i, 4}, {j, 4}]], {k, 0, 2, 2}]
  cl[3] = Table[Coefficient[If[Length[cs[i, j]] = 4, cs[i, j, 4, 0] L, {i, 4}, {j, 4}];
  c[3] = Table[Coefficient[If[Length[cs[i, j]] = 4, cs[i, j, 4, 0] L, {i, 4}, {j, 4}];
  cCon[3] = Simplify[Table[cs[i, j, 4, 0] L - cs[3][i, j] s, {i, 4}, {j, 4}]];]
  cTemp = cs / . L → cCross;
  cl[1] = Table[Coefficient[cTemp[i, j, 2] L, {i, 4}, {j, 4}];
  c[1] = Table[Coefficient[cTemp[i, j, 2] s, {i, 4}, {j, 4}];
  cCross[1] = Table[Coefficient[cTemp[i, j, 2], cCrossTerm, {i, 4}, {j, 4}];
  cCon[1] = Simplify[Table[cTemp[i, j, 2] = (cl[1][i, j] L + cs[1][i, j] s + cCross[1][i, j] cCrossTerm), {i, 4}, {j, 4}]];]
  ```
The Gap Functions Derived and Analyzed

We seek an upper bound on the maximum absolute value of any entry in $K - I$. Start with $K - I$ and turn all coefficients positive.

$$\text{raw0} = \text{Expand}\left[\sigma \left(\text{Row}\left[\begin{array}{cccc}
\text{cCon[0]} & \text{cCon[1]} & \text{cCon[2]} & \text{cCon[3]}
\end{array}\right], 
\text{L}\end{array}\right) + \text{Row}\left[\begin{array}{cccc}
\text{L}\end{array}\right]
\right]
$$

\[
\text{raw0} = \begin{array}{cccc}
6 & 25 & 216 & 6 - 39 \\
25 & 50 & 31 & 1 - 31 \\
3 & 25 & 10 & 1 - 2 \end{array}
\]

Get everything in terms of $L$. 

$$H[1] = \text{cCon[1]} + \text{cCon[1]} L + \text{cCon[1]} L + \text{cCross[1]} L = \text{Do}[H[1] = \text{cCon[1]} + \text{cCon[1]} L + \text{cCon[1]} s, \{i, \{0, 2, 3\}\}]
$$

matrixK = K; \text{raw} = 3125 (K - I) so \text{raw} / 3125 + I = K; KCheck is defined from the coefficient matrices $H$, and adjusted in the same way. All three agree,
raw1 = Apart[Simplify[raw0 //. \{\sigma \to \text{Sin}\left[\frac{\delta}{2}\right], \text{Sin}\left[\frac{\delta}{2}\right] \to \frac{2 \pi}{\sqrt{5}\ (L + 1)}\}]]

raw1[\{1, 1\}]

\[
\begin{bmatrix}
6 & 3 & 42 \ L^3 & 18 \ L^3 & 126 \ L^7 & 9 \ L^2 & 42 \ L^3 & 12 \ L^3 \\
125 \ (L + 1)^8 & 125 \sqrt{5} \ (L + 1)^8 & 125 \ (L + 1)^8 & 125 \sqrt{5} \ (L + 1)^8 & 125 \ (L + 1)^8 & 25 \sqrt{5} \ (L + 1)^8 & 25 \ (L + 1)^8 & 25 \sqrt{5} \ (L + 1)^8
\end{bmatrix}
\]

All terms are like the preceding, with \(L\) in the numerator and a power of \(L + 1\) in denominator. Next we turn \(\frac{1}{L+1}\) into \(\frac{1}{L}\), since we are chasing an upper bound.

raw2 = raw1 //. \(1 + L\)^\(-n\) \to \(L^n\);

raw2[\{1, 1\}]

\[
\begin{bmatrix}
6 & 3 & 42 & 18 & 126 & 9 & 42 & 12 & 42 & 9 \\
125 \ L^8 & 125 \sqrt{5} \ L^8 & 125 \ L^8 & 125 \sqrt{5} \ L^8 & 125 \ L^8 & 25 \sqrt{5} \ L^8 & 25 \ L^8 & 25 \sqrt{5} \ L^8 & 25 \ L^8 & 9
\end{bmatrix}
\]

Then we use the assumption \(L > \frac{3}{\epsilon}\) and we bound \(\frac{1}{L}\) by \(\frac{\epsilon}{2}\), which we accomplish by replacing \(L\) by \(\frac{2}{\epsilon}\).

raw3 = Expand[raw2 //. \(L + \frac{2}{\epsilon}\) \to \(L\)];

raw3[\{1, 1\}]

\[
\begin{bmatrix}
3 \pi^2 \ \epsilon^2 & 21 \ pi^2 \ \epsilon^2 & 3 \ pi^4 \ \epsilon^2 & 63 \ pi^2 \ \epsilon^3 & 9 \ pi^4 \ \epsilon^3 & 21 \ pi^2 \ \epsilon^4 & 9 \ pi^4 \ \epsilon^4 & 3 \ pi^4 \ \epsilon^4 & 21 \ pi^2 \ \epsilon^5 \\
125 & 250 & 500 \sqrt{5} & 500 \sqrt{5} & 200 & 400 \sqrt{5} & 625 & 400 & 3 \ pi^2 \ \epsilon^5 & 6 \ pi^4 \ \epsilon^5 & 9 \ pi^4 \ \epsilon^5 & 6 \ pi^4 \ \epsilon^6 & 9 \ pi^4 \ \epsilon^6 & 27 \ pi^4 \ \epsilon^6 & 9 \ pi^6 \ \epsilon^6 & 21 \ pi^2 \ \epsilon^7 & 9 \ pi^4 \ \epsilon^7
\end{bmatrix}
\]

Here is a quick approximation.

Collect[\[
\begin{bmatrix}
4 \_raw3[\{1, 1\}, \epsilon] / (\{zz_\ \epsilon^k \mapsto N\[zz]\] \ \epsilon^k, \ \epsilon \mapsto N\[zz]\] \ \epsilon\}/.
\{\_?NumericQ \ \epsilon^k -> Ceiling[100 \ x] / 100., \_?NumericQ \ \epsilon^k -> Ceiling[100 \ x \ \epsilon^k] / 100.\}]
\]

0.95 + 3.62212 \ \epsilon + 6.42 \ \epsilon^2 + 6.53 \ \epsilon^3 + 4.25 \ \epsilon^4 + 1.45 \ \epsilon^5 + 0.64 \ \epsilon^7

The limit of the ratio to the desired \(\frac{\pi^2}{4} = \frac{125 \ \pi^2}{125} \) which is less than 1. And if \(\epsilon < \frac{1}{100}\), the \(\epsilon\)-expressions are all less than 0.9999. But these expressions are increasing polynomials that vanish at 0, so we live in \([0, 1]\) provided \(0 < \epsilon < \frac{1}{100}\).
\[
\text{Max}\left[\lim_{\epsilon \to 0} \frac{\text{raw3}}{\epsilon}\right]
\]
\[
\text{N}\left[\text{Max}\left[\lim_{\epsilon \to 0} \frac{\text{raw3}}{\epsilon}\right]\right]
\]
\[
\frac{12 \pi^2}{125} = 0.947482
\]
\[
\text{Max}\left[\frac{\text{raw3}}{\epsilon}, \epsilon \to \frac{1}{100}\right]
\]
\[
0.999898
\]

And in fact one could replace the 2 by \(\frac{24}{25} \frac{1}{\pi^2}\), or about 1.895; the limit above will be 1 in this case.

### The Vanishing-Gap Subsequence for the Quadrahelix

The denominators of the convergents for \(\frac{\theta}{2\pi}\), less 1, give the good \(L\)-values.

\[
\text{Denominator}\left[\text{Convergents}\left[\frac{\theta}{2\pi}\right]\right] - 1
\]
\[
\{0, 1, 2, 7, 10, 29, 40, 70, 182, 253, 1960, 12019, 13980, 26000, 143985, 601944, 5561490, 6163435, 11724926, 17888362, 65390015\}
\]

### The Limiting Angles

The dot product gives the cosine, the four angles sum to \(2\pi\), and alternate angles are equal, so this tells the whole story.

\[
L1 = \text{TrigExpand}\left[\text{ExpandAll}\left[\text{base2}[1] - \text{base1}[2]\right]\right];
\]
\[
L2 = \text{TrigExpand}\left[\text{ExpandAll}\left[\text{TrigExpand}\left[\text{base3}[1] - \text{base2}[1]\right]\right]\right];
\]
\[
dot = L1.L2;
\]
\[
\text{Limit}\left[\frac{\dot{\text{dot}}}{\sqrt{L1.L1} \sqrt{L2.L2}} /, L \theta \to \theta, L \to \infty\right]
\]
\[
\frac{1}{5}
\]

### The Limiting Rhombus

To get the rhombus we can take the first two points to be \((0, 0)\) and \((0, 1)\).

\[
\text{Clear}[A]; A[n_] := (x[n], y[n]);
\]
\[
\]
\[
\text{eqnsUnitLength} = \text{Table}\left[1 = (A[\text{Mod}[i + 1, 4, 1]] - A[i]).(A[\text{Mod}[i + 1, 4, 1]] - A[i]), \{i, 4\}\right];
\]
\[
\text{eqnsAngle} = \text{Table}\left[\frac{\{1\}^\text{T} = (A[\text{Mod}[i + 1, 4, 1]] - A[\text{Mod}[i + 1, 4, 1]]).\{A[\text{Mod}[i + 1, 4, 1]] - A[\text{Mod}[i + 1, 4, 1]]\}, \{i, 4\}\right];
\]
\[
A /\theta \text{Range}[4] /. \text{Solve}[\text{Join}[\text{eqnsUnitLength}, \text{eqnsAngle}], \text{Join}[A[3], A[4]]]\]
\[
\left\{\{0, 0\}, \{0, 1\}, \left\{\frac{2 \sqrt{6}}{5}, \frac{4}{5}\right\}, \left\{\frac{2 \sqrt{6}}{5}, \frac{1}{5}\right\}\right\}, \left\{\{0, 0\}, \{0, 1\}, \left\{\frac{2 \sqrt{6}}{5}, \frac{4}{5}\right\}, \left\{\frac{2 \sqrt{6}}{5}, \frac{1}{5}\right\}\right\}
\]

### The Vanishing-Gap Subsequence for the Octahelix

Here are some examples of how \text{LatticeReduce} yields good \(L\)-values for the octahelix.
\( \gamma = \text{ArcCos} \left[ \frac{1}{12} \left( -3 + \sqrt{3} \right) \right] ; \gamma_1 = \text{ArcCos} \left[ \frac{1}{12} \left( -3 - \sqrt{3} \right) \right] \); \( s = 10^{10}; w = \sqrt{s} \);

\( (a, b, c) = \text{Round} \left[ s \left( \theta, 2 \pi, -\gamma \right) \right] \);

MatrixForm\(\{A = \{(a, 1, 0, 0), (b, 0, 1, 0), (-c, 0, 0, w)\}\}\)

\(\begin{bmatrix}
23005.23983 & 1 & 0 & 0 \\
62831.853072 & 0 & 1 & 0 \\
10795.923711 & 0 & 0 & 100000
\end{bmatrix}\)

MatrixForm\(\{B = \text{LatticeReduce}[A]\}\)

\(\begin{bmatrix}
-14105 & -64708 & 23692 & 100000 \\
45507 & 79278 & -29027 & 100000 \\
-365998 & 169987 & -62239 & 0
\end{bmatrix}\)

The first vector is the negative of what we want.

\(\text{N[Mod}[64708 \theta - \gamma, 2 \pi, -\pi]]\)

\(3.30883 \times 10^{-8}\)

Sometimes the requirement that \(l \) be positive means that the value must be adjusted by 1 and works for \( \gamma^- \) as opposed to \( \gamma^+ \). Here is an example.

\(s = 10^{200}; w = \sqrt{s} \);

\( (a, b, c) = \text{Round} \left[ s \left( \theta, 2 \pi, -\gamma \right) \right] \);

\(A = \{(a, 1, 0, 0), (b, 0, 1, 0), (-c, 0, 0, w)\}\);

\(\text{N[MatrixForm}\{B = \text{LatticeReduce}[A]\}\}\)

\(\begin{bmatrix}
-1.61399 \times 10^{100} & 5.21269 \times 10^{98} & -1.90857 \times 10^{98} & 0. \\
7.00621 \times 10^{99} & 1.00217 \times 10^{100} & -3.66936 \times 10^{99} & 1. \times 10^{100} \\
-6.95597 \times 10^{99} & 1.8658 \times 10^{100} & -6.83144 \times 10^{99} & -2. \times 10^{100}
\end{bmatrix}\)

\(\text{n = Select[}\{B, \text{Last}\[n]\}\}[[1, 2]]\)

\(\text{SMaxExtraPrecision} = 1000; \)

\(\text{N[Mod[(n - 1) \theta - \gamma_1, 2 \pi, -\pi], 20]}\)

\(10021.748859 140.317070 0.670 606.276 0.35026.550 \times 10^{0} 890 854 358 977 791 419 357 055 984 002 968 975 063 788 933 115 779 224 144.920 190\)

\(4.302462702862659767 \times 10^{-101}\)