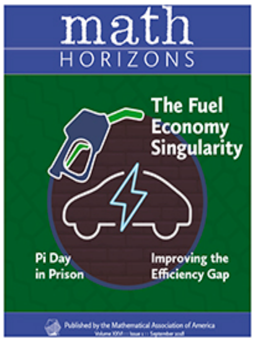

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Resolving the Fuel Economy Singularity

Stan Wagon

Consider this classic puzzle: Charlie runs four miles at A miles per hour and then four miles at B mph. What is his average speed for the eight miles?

At a glance, this puzzle seems the same as this problem: Diane runs at a rate of C minutes per mile for the first four miles and D minutes per mile for the second four. What is her average rate?

But this second average is easy to compute using the familiar *arithmetic mean*: $(C+D)/2$. You can check that Charlie's average speed is the *harmonic mean* of A and B,

$$\frac{2}{\frac{1}{A}+\frac{1}{B}}.$$

This idea applies to other rates as well. Suppose a family has two vehicles, one getting 20 miles per gallon (MPG) and the other 40 MPG, and they each cover the same distance in a year. Then the combined fuel economy is the harmonic mean of 20 and 40, 26.7 MPG.

I recently purchased a Chevrolet Bolt, an electric vehicle (EV) with a 60 kilowatt-hour (kWh) battery and a 238-mile range, and realized that this unexpected behavior of rates has surprising ramifications in measuring energy economy—not only for gasoline-powered cars, but also especially for EVs. The new feature of EVs (also hybrids) is that energy consumption can be both positive and negative.

The MPG Illusion

American drivers are familiar with the MPG measure of fuel efficiency; 40 MPG is a typical value. Other countries use liters per 100 kilometers (40 MPG is 5.9 L/100km). The choice of units is not important, but the question of which unit to put in the denominator is.

Using MPG to measure fuel economy can skew our thinking in terms of improved efficiency. Consider this scenario: Alice buys a new SUV, improving her usage rate from 15 to 20 MPG, while Bob trades in his car, improving his usage rate from 40 to 50 MPG. Who will save more fuel? Alice will save 1.67 gallons for 100 miles ($\frac{100}{15} - \frac{100}{20} = \frac{5}{3}$), whereas Bob saves only a half-gallon over the same distance (figure 1). In order to match Alice's gain, Bob would need a vehicle getting 120 MPG. This phenomenon, caused by ignoring the complexity of reciprocals, is known as the *MPG illusion* (see Richard Larrick and Jack Soll, "The MPG Illusion," *Science* 320 [June 20, 2008]: 1593–1594 or *www.mpgillusion.com/p/what-is-mpg-illusion.html*).

The difference between Alice's and Bob's upgrades is much easier to see when the distance is in the denominator: Using the gallons-per-100-miles (which we abbreviate to GPM) measure, Alice improves from 6.67 to 5 GPM, while Bob improves from 2.5 to 2 GPM.

For a more extreme example, if Alice has a truck, which she improves from 10 to 14 MPG, and Bob has a 40-MPG car, it is impossible for him to upgrade his car to match her improvement. Even a perpetual motion vehicle using no fuel would not match the truck's fuel savings!

Larrick and Soll wrote, "Relying on linear reasoning about MPG leads people to undervalue small improvements on inefficient vehicles. We believe this general misunderstanding of MPG has implications for both public policy and research on environmental decision making." Because of these issues, the label placed by the Environmental Protection Agency on new cars now shows GPM and MPG.

MPG is indeed useful for measuring range and is in no danger of being abandoned. But the MPG illusion is only one of several serious problems with putting fuel in the denominator.

The MPG Paradox

The potential to negatively affect public policy decisions is not merely hypothetical. The government imposes penalties for automakers whose fleet doesn't satisfy the Corporate Average Fuel Economy (CAFE) standards. Steven Tenn and John Yun observed that the penalty for a carmaker might *increase* when it adds a fuel-efficient car to the fleet ("When Adding a Fuel Efficient Car Increases an Automaker's CAFE Penalty," *Managerial and Decision Economics* 26, no. 1 [2005]: 51–54). This surely is not the intent of the law; it appears that the people who devised the penalty formula were blinded by the traditional reliance on MPG and did not understand how fuel economy really works.

Suppose the government target is T MPG and a manufacturer makes n cars of several models having an average fuel economy of H MPG, which was computed (correctly) using the harmonic mean. Then the CAFE penalty is $Pn(T - H)$, where P is the penalty per car, in dollars.

Here is an example of the paradox that arises from the use of the MPG-based penalty. Suppose a company makes 1,000 cars, each of which gets 30 MPG. Suppose the government target is 50 MPG and the penalty coefficient is \$50. Then the penalty is $\$50(1{,}000)(50 - 30) = \$1{,}000{,}000$. Now suppose the fleet is enhanced by the addition of a single new car attaining 70 MPG. This improves the average fuel economy to

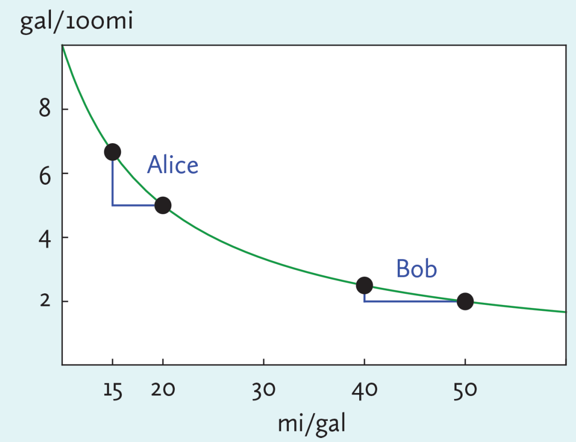
$$\frac{1{,}001}{\frac{1{,}000}{30} + \frac{1}{70}} = 30.017 \text{ MPG},$$

but it does not offset the fact that a penalty must be paid on all cars. The penalty is now \$1,000,142.

Quoting Tenn and Yun, "This result is driven by the fact that CAFE's regulatory tax punishes fuel efficient, as well as inefficient, vehicles. With such a blunt regulatory instrument, it is not surprising that CAFE can create peculiar incentives."

A better alternative is the penalty $P_0 \sum (G_i - T_0)$, where T_0 is the GPM target and G_i is the GPM rating of the ith car (Carolyn Fisher, "Let's Turn CAFE Regulation on Its Head," *Resources for*

Figure 1. The relation between MPG and GPM ($y = 100/x$). Bob's Δx is twice Alice's but his Δy (the fuel savings) is less than one-third of Alice's.



the Future [2009], *http://bit.ly/improvedCAFE*). This approach treats each car as a separate entity. The preceding example becomes $T_0 = 2$, $P_0 = \$750$, $n = 1{,}000$, and G_i is 100/30 for each noncompliant car and 100/70 for the new car. The efficient car reduces the total penalty by $\$750(2 - \frac{100}{70}) = \$429$.

The MPG Singularity

When coasting in a gasoline-powered car, fuel consumption can be zero; division by that number means that the instantaneous MPG is infinite. The solution of using a large number to replace ∞ is often used, but that causes additional problems in EVs that wish to present charts, not just single numbers. In an EV, fuel is measured in kWh and consumption in miles per kWh (similar to MPG) or kWh per 100 miles (similar to GPM). The Bolt presents the driver with a screen showing energy consumption over the past 50 miles and the Tesla S does the same over 30 miles.

A driver needs to know the average fuel consumption during a trip, as it is critical that he or she get home (or to a charging station) before the battery dies. In my case, with a 60-kWh battery, if I start out on a 230-mile drive I want to be sure that my average consumption is very close to 4 miles per kWh (mi/kWh).

Let $k(x)$ denote the kWh used in the first x miles of a trip. Figure 2 shows a graph of $k(x)$ in which there is a long downhill between miles 12 and 18.5; on that stretch, energy regeneration decreases the total energy consumed.

The Tesla and Bolt do not show the graph in figure 2. Tesla's display resembles figure 3; it shows the consumption rate—the derivative $k'(x)$. It also shows the average rate, $k(30)/30$ (multiplied by 1,000 for Wh instead of kWh), as a number and as a dashed line. The units, Wh/mi, are analogous to the GPM measure.

The Bolt uses the MPG-like form mi/kWh and therefore uses the reciprocal, $1/k'(x)$. But the reciprocal creates a big problem. Because k' can be 0—this happens twice in the data of figures 2 and 3—we can have singularities, as shown in figure 4.

Figure 2. Energy used on a trip with a long downhill starting 12 miles into the trip.

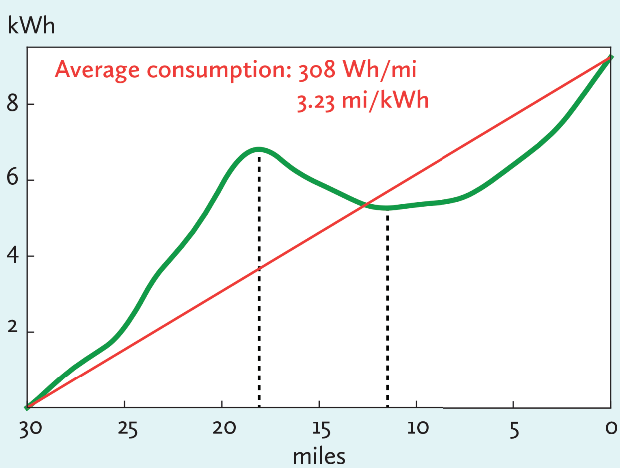


Figure 3. Tesla's consumption chart. The dashed line is the average; the solid line is the EPA rating of 350 Wh/mi. The green region indicates regeneration of energy into the battery.

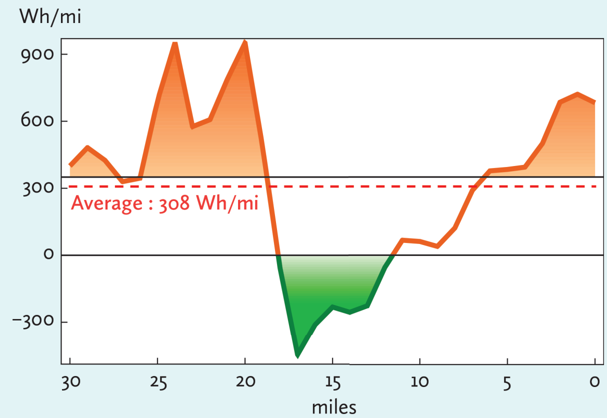
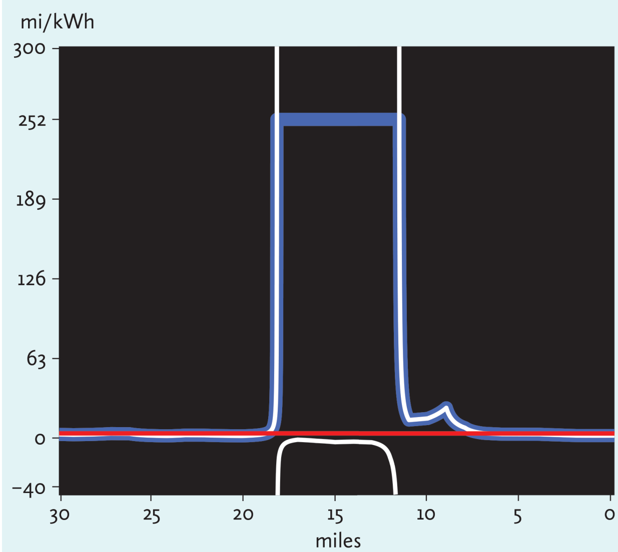


Figure 4. The graph of $1/k'$ is white, the average value (3.25 mi/kWh) is red, and the blue curve uses 252 as a proxy for any value that is negative or larger than 252.



The red line in figure 4 shows the overall average. Note that the average rate is a harmonic average of the graph. It is simply $30 / k(30) = 3.25$ mi/kWh, but if f is the function $1 / k'$, in figure 4, it is

$$\frac{1}{\frac{1}{30}\int \frac{1}{f}},$$

the continuous version of the harmonic mean. The precise location of the red line is impossible to discern in figure 4.

For a reason explained in the sidebar on page 5, Chevrolet wishes never to show negative numbers. They use the expedient of replacing any negative number or number larger than 252 with 252. That succeeds in eliminating the troublesome singularities! And in fact, the resulting graph (the blue line in figure 4) is not an unreasonable approximation to the true data.

The Bolt does not show the graph in figure 4; instead, it displays a bar chart based on five-mile intervals as in figure 5. Each bar height in the bar chart can be computed exactly as, for example, $5 / (k(20) - k(15))$. But the move to the bar chart does not address the problem of the impossible-to-interpret red line. This is inevitable, given the choice of units and the use of 252, because the chart will very often have a large vertical range.

For a bar chart based on k', the average is the arithmetic mean of the bar heights—or equivalently, their total area. But

MPG, MPGe, and MPG_{ghg}

The Environmental Protection Agency introduced MPGe so that one can evaluate an EV using the familiar MPG metric.

A gallon of gasoline has an energy content of 33.7 kWh. So if a car gets 3.97 mi/kWh (the Bolt's rating), that would translate to an MPG equivalent of $(3.97)(33.7) \approx 134$ MPG. But battery charging is not perfectly efficient: It takes about 1.12 kWh of charging electricity to put a kWh into the battery. Hence 3.97 miles per battery-kWh translates to 3.54 miles per purchased-kWh. So the MPGe is really $(3.54)(33.7) \approx 119$ miles per virtual gallon. This explains why the Bolt's range is given as 238 miles from the 60 kWh battery, but its MPGe, at 119, is much lower than the 238 value naively indicates.

It is important to compare vehicles in a more holistic way in terms of total production of greenhouse gases, both in manufacturing and driving. For an EV, a key issue is how the electricity is generated, and this varies tremendously by region. A comprehensive study introduced MPG_{ghg} to make this comparison (Rachael Nealer, David Reichmuth, and Don Anair's 2015 report "Cleaner Cars from Cradle to Grave," *www.ucsusa.org/EVlifecycle*).

For example, if electricity comes from burning natural gas, then an EV might rate 58 MPG_{ghg}, meaning that a traditional car getting 58 MPG would be equivalent to that electric car in terms of total greenhouse gas production. Such a rating depends on many things, such as how exactly the batteries and the car engine were manufactured, but the dominant factor is the source of the electricity.

Multiplication Is Not as Associative as We Think

We know that $\frac{-\text{mile}}{kWh}$ and $\frac{\text{mile}}{-\text{kWh}}$ are mathematically identical, but the perceived meanings are very different. A rate of five negative miles per kWh will be generally incomprehensible, eliciting responses such as, "What is a negative mile?" and "I'll never drive five miles in reverse." But "five miles gives one negative kWh" makes perfect sense: One must drive five miles to generate a stored kWh of energy.

Using reciprocal units, this rate is -0.2 kWh/mi and the meaning is clear: a fifth of a kWh of regeneration for each mile driven.

So while we take multiplicative associativity for granted, it is not really as universal a law as we think when perception is taken into account.

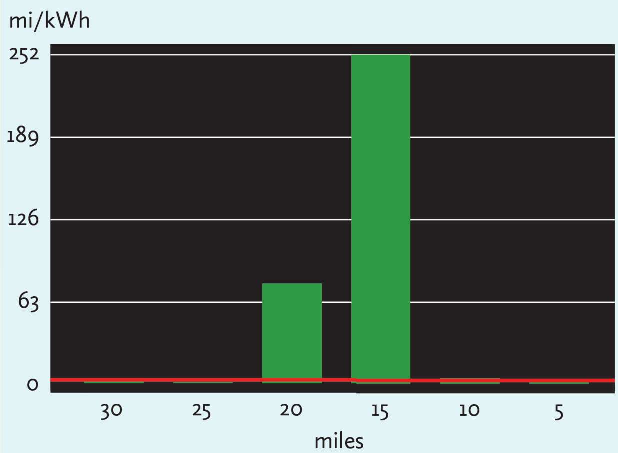


Figure 5. The Bolt's bar chart reduces the data of figure 4 to five-mile intervals. The red line is the average.

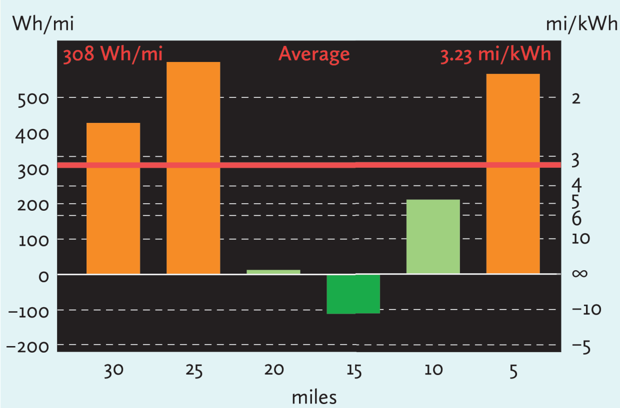


Figure 6. The consumption rate shown by a method that avoids the singularity and has a reasonable scale in all situations.

because the Bolt's chart uses $1/k'$, the average is approximated by the harmonic mean of the heights. So it has little connection to the area of the bars.

In fact, the Bolt's designers do not place the average line at the correct value (red line in figure 5). Instead, the red line is shown at the average of the bar heights. Therefore, in the actual Bolt screen using the data of figure 5, the red line would be at 56.6 mi/kWh as opposed to the correct 3.23; this is a 1,652% error. Chevrolet did this deliberately on the assumption that the driver would want to see the average of the green bars, but the consequence is that the driver must deal with a very incorrect average.

I was able to arrange a conversation with some Bolt engineers and emphasized that one should not present wildly inaccurate data. I believe they will rethink this approach in future models. This example shows the difficulty of using a classic MPG-like scale in EVs.

Eliminating the Singularity

A solution to the singularity problem that works with both MPG and mi/kWh is to use a nonlinear labeling. For an EV, one can use a kWh/100mi scale for the bars, but with both units for the labels as in figure 6. Then one can read the average using either set of units (Wh/mi are used to avoid fractions).

An advantage of this approach is that the scale need never change. Going from −200 to 600 Wh/mi covers almost all situations. The singularity makes its presence felt in the ∞ that labels the horizontal axis; but it is of no consequence. In this scheme, lower means less energy consumption throughout the range and one can represent mi/kWh values of essentially any size without sacrificing legibility for the values of interest, typically between 2 and 6 mi/kWh. Another advantage of this method is that the average is located at the average of the bar heights and so is identical to what the driver senses it should be.

Both mi/kWh (likewise MPG) and kWh/100mi (GPM) have their place. Drivers in the United States will generally want to see the usage rate in mi/kWh. The chart of figure 6 allows this by making use of the simple scale of the kWh/100mi method. But the illusion, paradox, and singularity that arise from the popularity of the MPG system indicate that distance belongs in the denominator and that the GPM and kWh/100mi measures really are superior.

Stan Wagon *is Professor Emeritus of Mathematics at Macalester College. As founding editor of* UltraRunning *magazine, he learned a lot about the different means that arise when analyzing running speeds and times.*

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