Question

Given an *m*x*n* rectangular array of *mn* points from a square lattice, for what values of *m*, *n* is it possible to join the points by line segments to form a circuit that does not intersect itself and has no two consecutive colinear line segments?

Answer

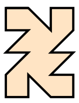
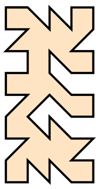
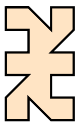
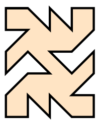
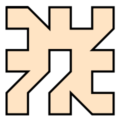
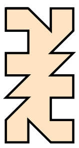
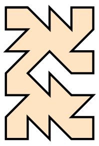
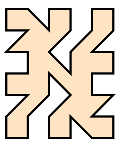
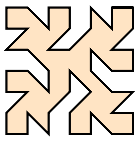
If *m* or *n* is = 1 or 3 it is impossible. If *m* or *n* but not both is 2, it is impossible. If *m*=*n*=5, it is impossible. All other cases are possible.

Possibility proof, constructive

We prove that for all values of *m*, *n* except those listed above, a circuit as specified exists, even with the further restriction that each line segment must have a length no greater than 2 (*i.e.* all lengths must be 1 or √2).

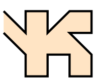
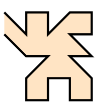
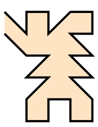
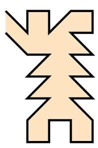
Step 1

All ten *m*x*n* grids in the table below can be solved, as seen in the ten diagrams.

	***m*=0 modulo 4**	***m*=1 modulo 4**	***m*=2 modulo 4**	***m*=3 modulo 4**
***n*=0 modulo 4**	 4x4			
***n*=1 modulo 4**	 4x5	 5x9		
***n*=2 modulo 4**	 4x6	 5x6	 6x6	
***n*=3 modulo 4**	 4x7	 5x7	 6x7	 7x7

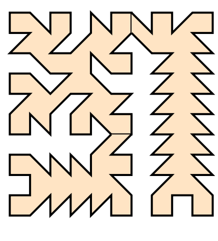
Step 2

For any of the ten diagrams above, we can append to its right side a 4x*n* diagram as shown below, to get an (*m*+4)x*n* diagram of a circuit that also meets the terms of the puzzle, while having no edge of length greater than 2.

***m*=4**	***m*=5**	***m*=6**	***m*=7**	***m*>7**
				**etc.**

We can also do this appending to a diagram that has been rotated, and to one that is the result of another such appending process. Thus, using induction, we can obtain a *m*x*n* circuit for any *m*>=4, *n*>=4, except 5x5.

So, for example, we can provide a solution for an 11x11 grid by starting with a 7x7, appending a 4x7, rotating the result clockwise, and appending an 4x11, as shown to the right.



We can also easily obtain a circuit for the only other possible grid, 2x2:  .

Impossibility of other cases

I don't have formal poofs of impossibility for the other cases. I offer ways to convince yourself that they're not possible:

For 1x*n*, any two consecutive edges are collinear, and so not allowed.

For 2x*n*, you can either draw a square, the possible 2x2 case; or you can draw a path which goes along the strip but can never reconnect to its start.

For 3x*n*, there are several ways to draw two paths which both run along the strip, between them visiting all the vertices. But you can't get them to join up at an end — consider what the middle vertex at an end can be connected to. (For a 3x*n* band, 3 wide and *n* long, any even *n* is possible. For a 3x*n* Möbius strip, any *n* not divisible by 8 is possible.)

For a proof that 5x5 is impossible, see
S. Chow, A. Gafni, and P. Gafni, *Connecting the dots: Maximal polygons on a square grid*, Mathematics Magazine. 94:2 April **2021**, 118-124.