Problem 1271.

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Represent any round-robin schedule as a 4x7 matrix, where the rows represent the four courts and the columns the seven days of the tournament. The following analysis achieves the goal of determining all possible solution structures, but not a goal of cataloguing all possible solutions to the problem. (There is some question as to when two matrix schedule solutions should be considered distinct. It can reasonably be assumed that any two matrix schedule solutions for which the (i,j) elements in each differ for at least one pair (i,j) are distinct.) The “structure” of the schedule matrix refers to the set of its “row types,” each row being one of six possible types—A, B, C, D, E, and F—defined later.

Number the eight players 1, 2, … , 8 and let them be the eight vertices in the complete graph *K*8. The tournament involves 7- coloring the edges of *K*8. Each column in the schedule matrix corresponds to one of the seven colors, with each player-pair of a given color assigned to one of the four courts. Use the notation *Pm* for a path with *m* vertices and *Cn*for a cycle with *n* vertices.

To avoid having any row with a player appearing three or more times, six of the players must appear twice and two must appear only once. (Moreover, the two that appear once must appear twice in every other row.) Thus, for any given row, the two player numbers appearing only once must form the endpoints of a path in *K*8 and the remaining player numbers must form a cycle or cycles in *K*8. Because *P*1, *C*1, and *C*2 are not possible, the six possible structures for a row in the schedule matrix are:

A: *P*8

B: *P*5 and *C*3

C: *P*4 and *C*4

D: *P*3 and *C*5

E: *P*2 and *C*6

F: *P*2 and *C*3 and *C*3

To find all possible solution structures, but not all possible solutions, create a computer program to start with a specification of the first row of the schedule matrix and then loop through all possibilities for the remaining rows, one row at a time, column by column, making certain to eliminate all candidates that do not satisfy the following constraints: (1) all eight player numbers must appear once and only once in each column , (2) no player number can appear more than twice in any row, and (3) all 28 player-pairs must appear in the schedule matrix.

The following numbers of solutions are found.

first row type A (1-2, 2-3, 3-4, 4-5, 5-6, 6-7, 7-8): 1200 solutions

first row type B (1-2, 2-3, 1-3, 4-5, 5-6, 6-7, 7-8): 432 solutions

first row type C (1-2, 2-3, 3-4, 1-4, 5-6, 6-7, 7-8): 0 solutions

first row type D (1-2, 2-3, 3-4, 4-5, 1-5, 6-7, 7-8): 840 solutions

first row type E (1-2, 2-3, 3-4, 4-5, 5-6, 1-6, 7-8): 1440 solutions

first row type F (1-2, 2-3, 1-3, 4-5, 5-6, 4-6, 7-8): 2016 solutions

Analyzing the various solutions leads to the following conclusions about the structure of a schedule matrix that solves Problem 1271.

1. There are solutions with all four rows type A: i.e. AAAA.
2. There are solutions with three rows type A and the other row type B or type D or type E or type F: e.g. AAAB, AAAD, AAAE, AAAF.
3. There are solutions with two rows type A and the other two rows one of the following pair-types: BD, BE, and EF: e.g. AABD, AABE, AAEF.
4. There are no solutions with only one row of type A.
5. There are solutions with no row of type A; each such solution has either (i) three rows type B and one row type E or type F or (ii) three rows type D and one row type E or type F: e.g. BBBE, BBBF, DDDE, DDDF.
6. There are no solutions with a row of type C.
7. The only possible solution structures are specified in 1, 2, 3, 4, 5, and 6 in this list.

A few comments stemming from conclusion 1: Conclusion 1 states that there are solutions for which each row is a path *P*8 in which the seven edges are each of a different color. Moreover, each row is a Hamiltonian path in *K*8. So, another way to state conclusion 1 is that there are solutions to Problem 1271 in which all 28 edges in a proper 7-coloring (edge coloring) of *K*8 are “covered” by four 7-colored (edge-colored) Hamiltonian paths. Perhaps surprisingly, this is NOT the kind of result one finds for Problem 1271 when posed with respect to six players on three courts in five tournament days. For that problem, there are no solutions with a *P*6 Hamiltonian path in *K*6. The only solutions have all three rows consisting of *P*2 and *C*4 and all three *P*2 have the same edge color. There are no solutions with *P*3 and *C*3. **This latter observation suggests an interesting question: Is there a solution to problem 1271 when posed for a round robin tournament with 2*n* players (*n*>2) on *n* courts played over 2*n*–1 days in which some row has the structure *Pn* and *Cn*?**