A Rolling Square Bridge: Reimagining the Wheel

Alfred Jacquemot, Price & Myers, London, UK; <alfred.jacquemot@gmail.com>

Thomas Randall-Page, Architectural Association, London, UK; <tomrandallpage@hotmail.com>

Antonín Slavík, Faculty of Mathematics and Physics, Charles Univ., Prague, Czech Republic; <slavik@karlin.mff.cuni.cz>

Stan Wagon, Macalester College, St. Paul, MN, USA; <wagon@macalester.edu>

1. Introduction

In 1960 G. B. Robison [6] discovered that a square can roll smoothly on a road made of linked catenaries. *Smoothly* here means that the center of the square moves only horizontally as the square rolls. Because the center moves neither up nor down, the rolling motion is in all aspects the same as a round wheel rolling along a straight line. Remarkably, the basic idea that makes this work—the fact that an infinite straight line can roll smoothly along the catenary $y = -\cosh x$, was proved and published by James Clerk Maxwell [3] over a century earlier. In 1997, Stan Wagon built a tricycle with square wheels at Macalester College (often called a square-wheel bicycle: the third wheel allows it to stay upright when at rest) that can be ridden smoothly on a 25-foot road (Fig. 1). This whimsical idea received a lot of publicity. But in 2022 a team led by architect Thomas Randall-Page and structural engineer Alfred Jacquemot took the concept to the next level: they built a steel bridge in London (Fig. 2) that is anchored by two squares, which can be rolled by hand so that the bridge turns upside-down allowing boats to pass.

Figure 1. Stan Wagon on his square-wheel bike (it is really a tricycle).

Figure 2. The rolling square bridge at Cody Dock in East London, England. Photo by G. G. Archard.

2. An Elegant Equation: $x'(\theta) = r(\theta)$

Suppose a wheel is given in polar coordinates as $r = r(\theta)$, with the origin viewed as the center of mass, and the corresponding road (i.e, the road on which the wheel will roll with the center of mass moving neither up nor down) is given in parametric form as $(x(\theta), y(\theta))$, the point on the road that is touched by the wheel point defined by θ. See Figure 3 and note that $x(-\pi/2) = 0$ and, typically, $y(\theta)$ is negative. We assume throughout that friction prevents any sliding.

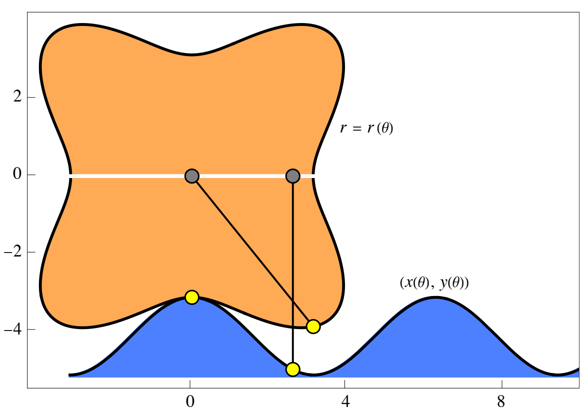

Figure 3. The relationship between a wheel and the corresponding road. The lengths of the black lines are equal; the length of the road between the two dots is the same as the length of the wheel between the two dots.

With this setup $\theta \mapsto (x(\theta), y(\theta))$ parametrizes the road. We have the *radius condition*: $y(\theta) = -r(\theta)$. Matching the arc length of the road to the relevant part of the wheel's perimeter (using the polar coordinate arc length formula) leads to: $\int_{-\pi/2}^{\theta} \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_{-\pi/2}^{\theta} \sqrt{r(\alpha)^2 + r'(\alpha)^2}\, d\alpha$. Differentiation with respect to θ and squaring gives $x'(\theta)^2 + y'(\theta)^2 = r(\theta)^2 + r'(\theta)^2$. The radius condition implies $y'(\theta) = -r'(\theta)$ and substituting this into the previous equality gives $x'(\theta)^2 = r(\theta)^2$, and consequently $x'(\theta) = r(\theta)$. This can be solved by the definite integral $x(\theta) = \int_{-\pi/2}^{\theta} r(t)\, dt$. The result can sometimes be inverted to get $\theta(x)$, the polar angle when the touching point is at x. When symbolic integration fails it is efficient to numerically solve the initial-value problem $x'(\theta) = r(\theta)$ and $x(-\pi/2) = 0$ to get a solution. For more on these formulas, and variants in the case that the wheel is given in a different form, or the case that the road is given and one seeks the appropriate wheel, see [2, 6, 8, 9, 11]; for a blog post with many videos and live demonstrations see [10].

3. The Catenary Road for a Straight Line

If $r(\theta) = 1$, the wheel is a circle, $x(\theta) = \theta + \frac{\pi}{2}$, and the road is the straight line $y = -1$. In 1849, James Clerk Maxwell, who was 18 years old when he published the result [3, p. 537], considered the case of a wheel that is the infinite straight line $y = -1$ with its center of mass taken to be $(0, 0)$. Maxwell's line was vertical; his exact words were:

"The straight line whose equation is $r = a \sec\theta$, rolled on a catenary whose parameter is a, traces a line whose distance from the vertex is a."

His "traces" refers to the locus of the origin, which is considered to be attached to the line. For our horizontal line, the polar form is $r = -\csc\theta$ (here $-\pi < \theta < 0$) and the road can be obtained by the integration $x(\theta) = \int_{-\pi/2}^{\theta} -\csc t \, dt = \ln\left(-\cot\left(\frac{\theta}{2}\right)\right)$. The parametric form of the road can be turned into the more familiar $y = f(x)$ as follows. The integration formula gives $\theta = -2\tan^{-1}(e^{-x})$ and so the road is given by $\csc\left(-2\tan^{-1}(e^{-x})\right)$, which is exactly $-\cosh x$, the formula for an inverted catenary. Figure 4 shows the rolling line. Maxwell did several dozen examples like this but he never considered truncating the line at a point where he could then make some 90° left turns to get a square wheel. It took over 100 years for someone to have that idea.

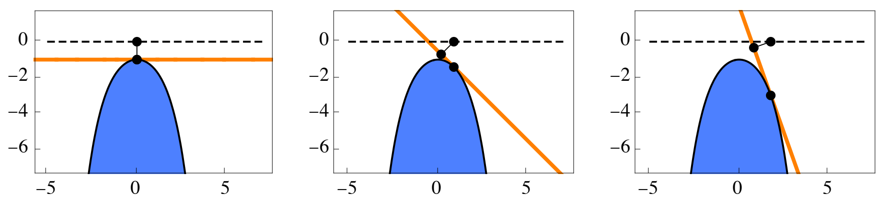

Figure 4. A line rolling along a catenary.

4. A Square Wheel

In 1960, G. B. Robison [6] rediscovered Maxwell's result and carried it farther to get a square wheel. His new idea was to truncate the catenary where its slope is ± 1. At those points the catenary makes a 45° angle with vertical, and so placing a another copy of the truncated catenary to its right gives a 90° cusp that is a perfect receptacle for the corner of a square (Fig. 5). Continuing yields an infinite bumpy road—though from the square's point of view it is not bumpy at all! The derivative of $\cosh x$ is $\sinh x$, so cusps occur when $x = \pm\sinh^{-1} 1$, about ± 0.88. To repeat the main point, the square will roll along this road with the center staying horizontal; therefore the movement does no work against gravity and is, in essence, no different than a circle rolling along a straight line.

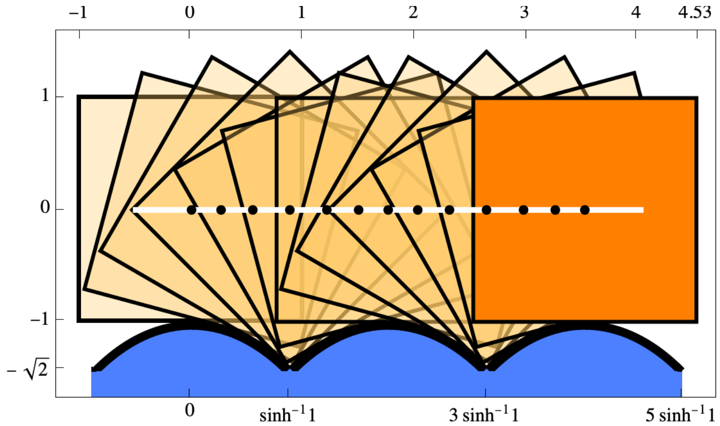

Figure 5. A square rolls smoothly along linked catenaries. The horizontal space needed to turn the square upside-down is $2 + 4 \sinh^{-1} 1$, or about 5.53.

A subtle point is the nonlinear relationship between x and θ (Fig. 6). This means that if one pedals at a constant angular rate the forward motion will not be at constant speed. But the relationship is sufficiently close to linear that the issue is not noticeable in practice.

Out[∘]=

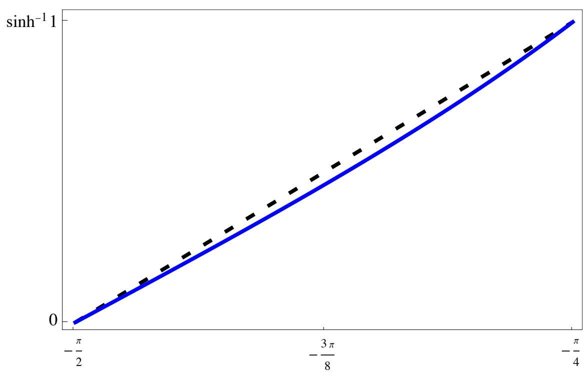

Figure 6. The x vs. θ relationship, which relates forward motion to rotation, is close to linear, but not exactly linear.

With this template in hand, one can construct the road and a square-wheel device that rides smoothly along the road. Figure 1 shows Stan Wagon on the tricycle at Macalester College that he designed in 1995; the road is 25 feet long. Figure 7 shows that Ripley's Believe It or Not found this whole idea hard to believe. And Figure 8 shows the round variant built by the National Museum of Mathematics in New York City; the museum has built copies of the device for science education organizations in Singapore, Ukraine, and Brazil.

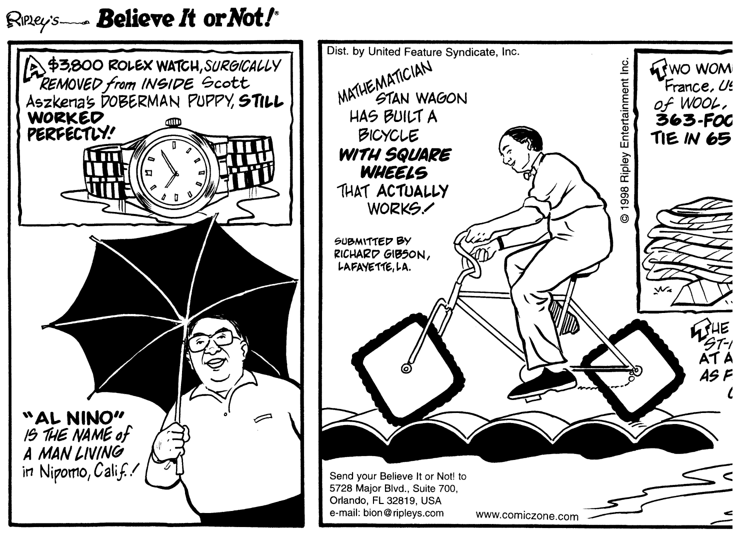

Figure 7. The square-wheel tricycle was featured in Ripley's *Believe It or Not*.

Figure 8. The circular version of the square-wheel tricycle at the National Museum of Mathematics in New York City. Photo by National Museum of Mathematics ⟨http://momath.org⟩.

There are two ways of building a simple model of a rolling square using, ironically, circles. A simple approach is to approximate the catenaries by quarter-circles of radius $r = 4/\pi$ and center $(0, -r - 1)$; these values mean that the circle and square have the same perimeter, so the square will roll without slipping (Fig. 9). Of course the center of the square will not stay horizontal; the maximum deflection is $\sqrt{2} - 1 - 2\left(2 - \sqrt{2}\right)/\pi$, about 0.041. The deflection indicates the extra work needed against gravity. The span of the road needed to turn the square upside-down is $8\sqrt{2}/\pi$, about 3.60; this is 2% greater than the catenary case $(4 \sinh^{-1} 1)$.

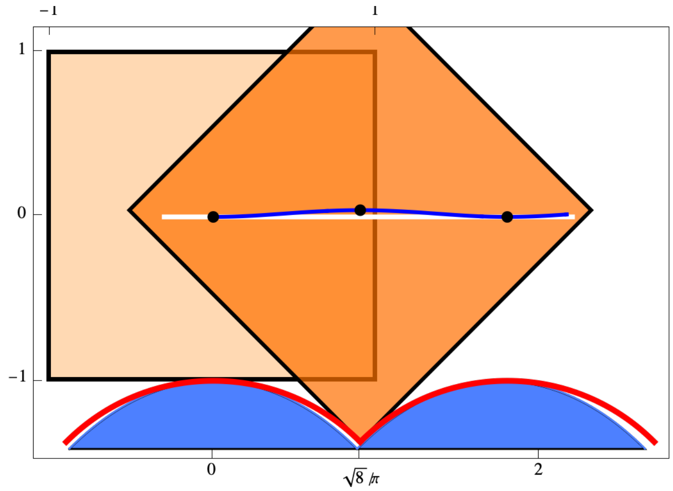

Figure 9. A square can roll on quarter-circles (red), but the square's center will wobble up and down. The ideal road—catenaries—is shown in blue.

A more interesting approach is to follow the instructions posted by the Exploratorium [12]; there it is described how round cylinders can be used to make a road on which a square can roll in such a way that it jumps over the cusp (Fig. 10). Easy geometry shows that, for a square of side-length 2, the radius of the circle should be $1/\left(\sqrt{2}-1+(\pi/4)\right)$. In this approach the corners of the square never touch the road. Again, the square's center moves up and down: the maximum deflection is 0.074. But the leap over the cusp means that the length of the part of the road that is in contact with the wheel to reach the upside-down position is quite small: about 2.62. Because friction at the contact point is a major contributor to the effort needed to roll the square, this is a big reduction in the amount of work. For the other methods the amount of contact is 4, the semi-perimeter. So this reduces the work against friction by about 33%. Also the span of the road needed to turn the square upside-down is about 3.33, compared to 3.53 for catenaries.

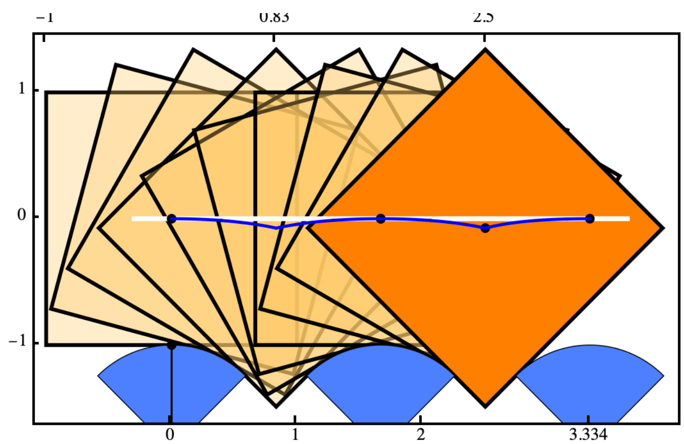

Figure 10. A square can roll on quarter-circles so as to avoid the cusp by jumping from one arch to the next. The center of the square will not stay horizontal.

The relatively simple fundamental equations allow one to investigate many road-wheel relationships (see [2]). One interesting result, discovered by Robison [6], is that there is only one wheel shape for which the road has the exact same shape as the wheel: the road is the parabola defined by $y=-x^2-\frac{1}{4}$ and the wheel is given by $y=x^2-\frac{1}{4}$ (see Fig. 11). To verify this, use the parabola's polar form $r=1/(2-2\sin\theta)$ and definite integration as discussed earlier. The function $x(\theta)$ then is $x=\frac{1}{2}\left(\cot\left(\frac{\theta}{2}\right)+1\right)/\left(\cot\left(\frac{\theta}{2}\right)-1\right)$.

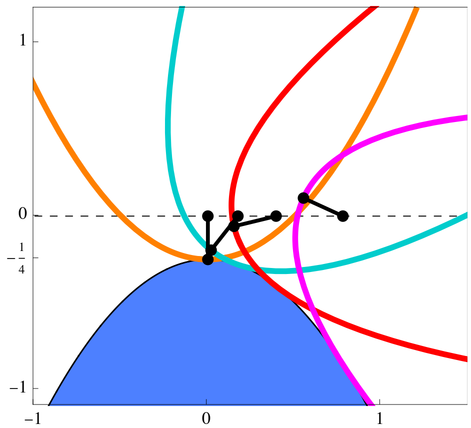

Figure 11. The parabola given by $y = x^2 - \frac{1}{4}$ rolls on a reflected copy of itself.

The case of a triangular wheel is interesting and will be discussed in §7. Wagon's post at [10] has a demonstration allows the reader to see the road–wheel relationship for various other shapes.

5. Bridge Design and Construction

Cody Dock, near where the River Lea empties into the Thames in East London, is being brought back into use after years of dereliction. The new bridge spans the dock mouth, allowing vessels to pass when it rolls along a track (Fig. 12) into an upside-down position. The bridge has a precise amount of counterweight in its upper structure so that the center of gravity is close to its geometrical center; this allows the 26,400-pound bridge to be moved by hand.

The idea behind the use of a rolling square arose from consideration of 19th century canal infrastructure and other preindustrial machines. Before steam engines and electrical power, all moving things required muscle, and thus balanced systems were used, often with counterweights to minimize the energy needed. It was with this in mind that Thomas Randall-Page designed this new type of opening bridge. Space was limited so a design was needed that did not occupy land on either bank. This led to a design that moved from the pedestrian mode to boat-passage mode along the channel rather than onto either bank.

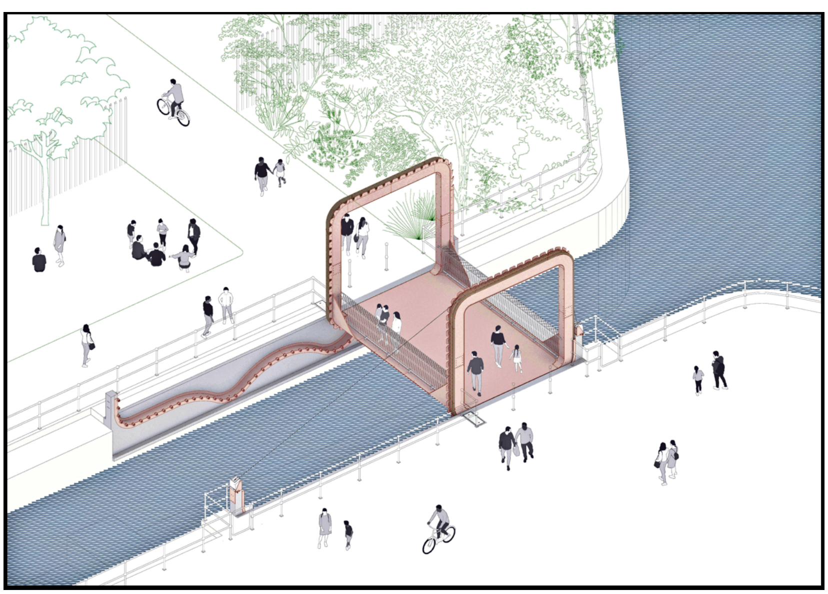

Figure 12. A sketch of the bridge with two winches shown on the near side. (Artwork by Thomas Randall-Page.)

The idea of a rolling square portal came from a basic knowledge that gears can be noncircular provided the offset from their pivot was mirrored from one gear to its partner. In exploring this idea the designer came across constructions, such as Wagon's square-wheel device, that forced the center of mass to move in a straight line. We are used to shapes such as squares and rectangles in our built environment and we are used to wheels being round. A rolling square is therefore unexpected and potentially unsettling (Fig. 13). Another factor was the hope that the rolling of the bridge would be performative and dramatic: not something one endured, but rather something that one eagerly anticipated.

Figure 13. The rolling square bridge in action. (Photo by Jim Stephenson.)

Once the basic principle was established, the main parameters were in place. The width of the channel was fixed. The designer (T. Randall–Page) and engineer (A. Jacquemot) needed to maximize the clear height for passing boats and this was limited by the length along the channel. In its pedestrian mode the bridge must align with the approach road and the existing dam limited the size of the squares. Given these limits, the key design tasks were:

1. The most significant and complex of these was understanding the abutment detail, where the square portal of the bridge interfaces with the track on the bank. This detail needed to deal with issues including:

- Maintaining a smooth motion while eliminating any chance of the bridge sliding on the track.

- Keeping the portals in line and in phase.

- Dealing with the wear of interfacing components, and designing in their various lifespans and replacement strategy.

- Accommodating thermal expansion of the bridge.
- Being able to see and remove debris on the track.
- Drainage of the track surface.

2. Designing a hand winch that would control the rolling of the bridge through 180° and be operated from just one bank of the channel. And a locking mechanism to keep the bridge stable in both the normal and inverted positions.

3. For the bridge to have a center of gravity very close to the geometric center, the weight of the deck had to be minimized while retaining torsional stiffness and spanning capacity in any orientation.

4. A lightweight rail was needed to act as a fall restraint, while also able to be folded down to give additional clearance for boats.

The footbridge is a simply supported structure with a monocoque steel deck spanning 7 meters. Two 5.5-meter rounded square portals at each end allow it to roll along a catenary track attached to concrete, which was in turn cast into the existing masonry walls. The upper section of each portal is counterweighted so that the center of gravity is raised to be close to the midpoint of the frame. The path geometry ensures that the midpoint stays horizontal when in motion so that the bridge weight is never lifted vertically (it is raised a little; see below). The bridge is moved by a cable attached to two manual winches on one bank. The handrails are constructed from a welded lattice of steel reinforcement bars and can fold down, via a torsional spring mechanism, for additional clearance when the bridge is inverted.

Like its Victorian forebears, the bridge design is tied to its functionality and its environment. Most of the structure is weathering steel, which has the desired strength, durability, and fabrication accuracy. Oak strips fixed to the hoops roll on the steel track, while precision-cut weathering steel teeth interlace with the steel pins. The rolling and guiding interfaces are kept separate, and the materials were chosen such that the softer component can be easily replaced within each interface, facilitating maintenance over its lifetime.

The geometry of the bridge track is based on the classic result that the road for a square wheel consists of inverted catenaries (§§3, 4). But the rounded corners, which were essential because of the gear teeth, required a new shape, and that was obtained by numerically calculating an elliptic integral, to get a curve that combines with the catenaries (see §6).

To predict the bridge behaviour during the roll, a staged analysis was carried out in parametric structural analysis software to assess how frictional effects affected the rotational and translational movement of the bridge. As the bridge is driven from one side only, ensuring adequate torsional stiffness was critical to prevent the portals skewing off course. Predictions were made for the frictional forces and resulting cable tensions and tested on site prior to the completion of the mechanical system design.

Monitoring the weight and geometry of the bridge was vital, in both design and construction. Any increase or offset in weight will increase frictional forces, which determine pin sizes, cable tensions, and ultimately the overall deck structure. These constraints lead to an inherently efficient structure. The counterweight uses a combination of concrete, denser steel plates, and rebar to ensure all the added weight is located at the highest points of the two squares. The internal ribs are hollowed-out where the stresses allow, and even the steel name plaques adorning each portal act as part of the counterweight. The overall center of mass is 50 mm below the geometric center. This was done to make it clear which winch is doing the work in each direction; if the true center was used, this point might be confused by a strong wind. So when the bridge is being rolled to its inverted position, the center of mass is lifted slightly (4 inches in 36 feet, about a 1% grade) and in the reverse direction it moves slightly downhill and one winch acts as a brake, helped by the inherent rolling friction of the system.

Despite the simplicity of this movement, the design process and fabrication revealed complex and unique engineering challenges. The constraints of the site, the geometry of the bridge movement and the forces in the bridge structure and cable are all carefully interlinked. This harmony had to be monitored at each stage, through design, fabrication, and installation.

6. The Mathematics of the Cody Dock Bridge

The Cody Dock bridge uses teeth and pins to guide the rolling steel structure, and for those to fit, the right-angles must be eliminated. The bridge accomplishes this by rounding the corners in the most natural way: using quarter-circles. Thus the wheel is the shape of the rounded square at left in Figure 14. Of course, the straight part of the square rolls on a catenary, but the rounded corner rolls on a curved shape that must be worked out precisely.

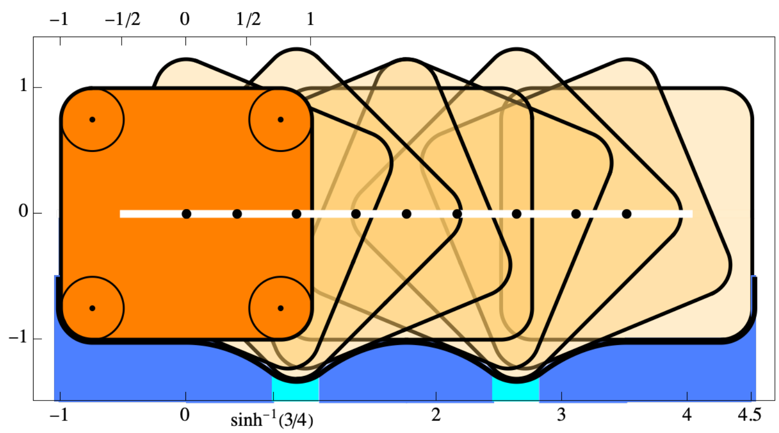

Figure 14. The wheel for the Cody Dock bridge is a square with rounded corners. The appropriate road consists of catenaries (blue), but an elliptic integral curve (cyan) is needed for the rounded corners.

To understand the complications caused by the rounded corners, it makes sense to study the more general problem of finding the proper road on which a circle will roll. Of course a normal circle with its center of mass at its center rolls on a straight line. But here we are using an abnormal circle: one whose center of mass is outside the circle. A similar setup occurs in the classic case of a rolling line in §3, where the center of mass of the linear wheel is well outside the wheel. The shape of the appropriate road turns out to be quite surprising. We will take the wheel to have radius 1 with its center of mass at the origin and geometric center at $(0, y_0)$, where $y_0 \leq -1$. Figure 15 shows the wheel centered at $\left(0, -3\sqrt{2}\right)$; this choice of y_0 corresponds to the Cody Dock case where the radius is $1/4$ and the square has side-length 2.

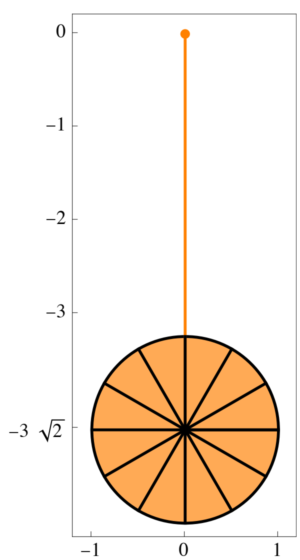

Figure 15. A wheel with center of mass at the origin and center $3\sqrt{2}$ below the origin.

Because the origin is outside the circle, it is not possible to describe the entire offset wheel in polar form. Such a wheel can be handled by representing it parametrically as $g(t) = (g_1(t), g_2(t))$, $t \geq 0$. Then, from [2], the parametrization of the road on which it will roll is

$$x(t) = \int_0^t \frac{g_1(s)\, g_2{}'(s) - g_1{}'(s)\, g_2(s)}{\sqrt{g_1(s)^2 + g_2(s)^2}}\, ds, \qquad y(t) = -\sqrt{g_1(t)^2 + g_2(t)^2}\,.$$

Our circular wheel corresponds to $g(t) = (\sin t,\, y_0 - \cos t)$, and therefore

$$x(t) = \int_0^t \frac{1 - y_0 \cos s}{\sqrt{1 + y_0^2 - 2\, y_0 \cos s}}\, ds, \qquad y(t) = -\sqrt{1 + y_0^2 - 2\, y_0 \cos t}\,.$$

To evaluate the definite integral, we find the corresponding primitive. The change of variable $s = \frac{\pi}{2} - u$ converts cosines to sines:

$$\int \frac{1 - y_0 \cos s}{\sqrt{1 + y_0^2 - 2\, y_0 \cos s}}\, ds = -\int \frac{1 - y_0 \sin u}{\sqrt{1 + y_0^2 - 2\, y_0 \sin u}}\, du.$$

Factoring $-y_0$ from the numerator and $-2\, y_0$ from the square root in the denominator, we get $\delta \int \frac{\beta + \sin u}{\sqrt{\gamma + \sin u}}\, du$, where $\beta = -\frac{1}{y_0}$, $\gamma = \frac{1 + y_0{}^2}{-2\, y_0}$, and $\delta = \sqrt{-\frac{y_0}{2}}$. To find a primitive for $\frac{\beta + \sin u}{\sqrt{\gamma + \sin u}}$, we split the fraction into $\frac{\gamma + \sin u}{\sqrt{\gamma + \sin u}}$, which is $\sqrt{\gamma + \sin u}$, and $\frac{\alpha}{\sqrt{\gamma + \sin u}}$, where $\alpha = \beta - \gamma$. Note that

$$\sqrt{\gamma + \sin u} \;=\; \sqrt{1 + \gamma - \left(\cos\left(\frac{u}{2}\right) - \sin\left(\frac{u}{2}\right)\right)^2} \;=\; \sqrt{1 + \gamma}\;\; \sqrt{1 - \frac{2 \sin\left(\frac{\pi}{4} - \frac{u}{2}\right)^2}{1 + \gamma}}\;.$$

Hence, we need

$$\sqrt{1 + \gamma} \int \sqrt{1 - \frac{2 \sin\left(\frac{\pi}{4} - \frac{u}{2}\right)^2}{1 + \gamma}}\; du + \frac{\alpha}{\sqrt{1 + \gamma}} \int \frac{1}{\sqrt{1 - \frac{2 \sin\left(\frac{\pi}{4} - \frac{u}{2}\right)^2}{1 + \gamma}}}\; du.$$

These primitives can be expressed in terms of the elliptic integrals, $E(\phi \mid m) = \int_0^\phi \sqrt{1 - m \sin^2 t}\; dt$ and $F(\phi \mid m) = \int_0^\phi \left(1 - m \sin^2 t\right)^{-1/2} dt$; we obtain

$$-2\, \sqrt{1 + \gamma}\; E\left(\frac{\pi}{4} - \frac{u}{2} \;\middle|\; \frac{2}{1 + \gamma}\right) + \frac{-2\, \alpha}{\sqrt{1 + \gamma}}\; F\left(\frac{\pi}{4} - \frac{u}{2} \;\middle|\; \frac{2}{1 + \gamma}\right).$$

Recalling that the primitive has to be multiplied by δ and that $u = \frac{\pi}{2} - s$, we find

$$\int \frac{1 - y_0 \cos s}{\sqrt{1 + y_0^2 - 2\, y_0 \cos s}}\, ds = 2\, \delta \left(\sqrt{1 + \gamma}\; E\left(\frac{s}{2} \;\middle|\; \frac{2}{1 + \gamma}\right) + \frac{\alpha}{\sqrt{1 + \gamma}}\; F\left(\frac{s}{2} \;\middle|\; \frac{2}{1 + \gamma}\right)\right) = (1 - y_0)\, E\left(\frac{s}{2} \;\middle|\; -\frac{4\, y_0}{(1 - y_0)^2}\right) + (1 + y_0)\, F\left(\frac{s}{2} \;\middle|\; -\frac{4\, y_0}{(1 - y_0)^2}\right).$$

One can use computer integration to avoid all this algebra. The following *Mathematica* code gets the primitive with no trouble.

```
FullSimplify[∫ (1 - y0 Cos[s]) / Sqrt[1 + y0^2 - 2 y0 Cos[s]] ds, y0 < -1]

-((-1 + y0) EllipticE[s/2, -(4 y0)/(-1 + y0)^2]) + (1 + y0) EllipticF[s/2, -(4 y0)/(-1 + y0)^2]
```

This primitive vanishes when $s = 0$. Therefore if we replace s by t, we get the definite integral from 0 to t, and the parametrization of the road is

$$x(t) = (1 - y_0)\, E\left(\frac{t}{2} \;\middle|\; -\frac{4\, y_0}{(1 - y_0)^2}\right) + (1 + y_0)\, F\left(\frac{t}{2} \;\middle|\; -\frac{4\, y_0}{(1 - y_0)^2}\right), \qquad y(t) = -\sqrt{1 + y_0^2 - 2\, y_0 \cos t}\,.$$

It is a bit of a surprise to learn that the road — the parametric plot of the preceding formula — is a sequence of loops (Fig. 16). It is therefore impossible to build a physical model for the entire wheel, but the bridge's corners require only a small piece of the loopy structure.

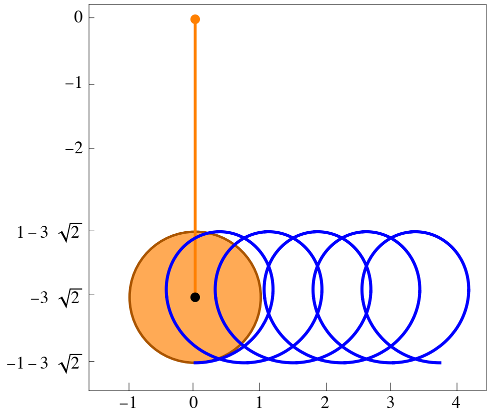

Figure 16. The loopy road for an offset circular wheel centered at $\left(0, -3\sqrt{2}\right)$. The center of mass at the origin oscillates right and left as the wheel moves through the loops.

Figure 17 shows how the offset circle rolls around the loops. It can be hard to visualize this motion, so in the figure $y_0 = -1.5$, which eliminates the intersections of the loops. For a full animation of this pendulum-like motion see [10]. The center of mass starts moving to the right, but then swings back to a point farther left than its starting position, then swings to the right, and so on. When y_0 is less than -2.41 the center of mass moves left of the y-axis during the first oscillation.

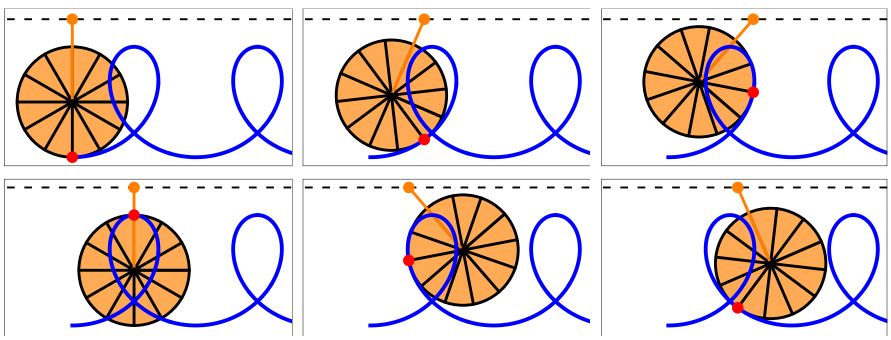

Figure 17. This sequence shows how the offset wheel moves pendulum-like as the the circle rolls around the loops. The center of mass at the origin oscillates right and left as the wheel rolls along the loops.

As y_0 approaches -1, the limit of the loopy curves is a sequence of semicircles. Strictly speaking, this curve is not a solution of our differential equation, because the denominator of the right-hand side vanishes at the cusps. And there is another plausible solution for $y_0 = -1$, a circle of radius 2. The wheel will roll inside this larger circle, and the center of mass oscillates back and forth along the x-axis; this setting is a special case of a hypocycloid where the stationary circle has twice the radius of the moving circle.

We now return to the bridge, a square with rounded corners; the road is a composite curve made by combining parts of two very different functions. The road starts with the familiar catenary, $y = -\cosh x$, to handle the straight part of the wheel. But when the rounded corner enters the picture, we need a part of the loop shown in Figure 16. Then of course another catenary, then another loop, and finally a catenary for the last step that completes the turning of the square upside down. Figure 18 shows how the parts of the two functions are combined to get the composite track that works for the bridge.

Out[ ]=

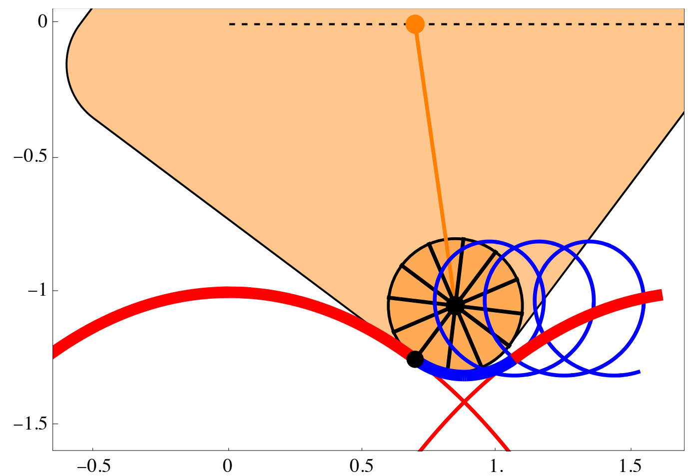

Figure 18. The track for the bridge (thick curves) combines the familiar catenary (red) and a small piece of the loopy road defined by elliptic integrals (blue).

Here is an interesting simplification. The preceding parametrization of the road for an offset round wheel used two elliptic integrals, E and F. Yet it is possible to express the part needed for the bridge using E only. The reason is that we can consider only the first corner, which is a quarter circle that is easily expressed in standard polar form. Take the width of the square to be 2 units and let b be the rounding radius. Easy geometry leads to this polar form for the lower right corner, where $\overline{b}$ denotes $1-b$, $B(b)=\cot^{-1}(\overline{b})$ and $-B(b)\leq\theta\leq B(b)-\frac{\pi}{2}$:

$$r(\theta)=\sqrt{2}\;\overline{b}\cos\left(\tfrac{\pi}{4}+\theta\right)+\sqrt{b^2-2\,\overline{b}^2\sin\left(\tfrac{\pi}{4}+\theta\right)^2}\,.$$

The initial straight part of the wheel has length $1-b$, and corresponds to $-\pi/2\leq\theta\leq-B(b)$. The corresponding part of the road, part of the catenary $y=-\cosh x$, ends in a point whose x-coordinate is $\sinh^{-1}(\overline{b})$. To get the next part of the road, we need to calculate

$$\begin{aligned}x(\theta)&=\sinh^{-1}(\overline{b})+\int_{-B(b)}^{\theta}\left(\sqrt{2}\;\overline{b}\cos\left(\tfrac{\pi}{4}+\rho\right)+\sqrt{b^2-2\,\overline{b}^2\sin\left(\tfrac{\pi}{4}+\rho\right)^2}\right)d\rho\\&=\sinh^{-1}(\overline{b})+\int_{-B(b)}^{\theta}\left(\alpha\cos\left(\tfrac{\pi}{4}+\rho\right)+b\,\sqrt{1-\beta^2\sin\left(\tfrac{\pi}{4}+\rho\right)^2}\right)d\rho\end{aligned}$$

for $\theta\in[-B(b),B(b)-\pi/2]$, where $\alpha=\sqrt{2}\;\overline{b}$ and $\beta=\sqrt{2}\;\overline{b}/b$.

Since $\cos\left(\frac{\pi}{4}+\rho\right)=\frac{\cos\rho-\sin\rho}{\sqrt{2}}$ has primitive $\frac{\sin\rho+\cos\rho}{\sqrt{2}}$ and $\sqrt{1-\beta^2\sin\left(\frac{\pi}{4}+\rho\right)^2}$ has primitive $E\left(\frac{\pi}{4}+\rho\,\middle|\,\beta^2\right)$, we get

$$x(\theta)=\sinh^{-1}(\overline{b})+\tfrac{\alpha}{\sqrt{2}}\,(\sin\theta+\cos\theta-\sin(-B(b))-\cos(-B(b)))+b\left(E\left(\tfrac{\pi}{4}+\theta\,\middle|\,\beta^2\right)-E\left(\tfrac{\pi}{4}-B(b)\,\middle|\,\beta^2\right)\right).$$

Simplifying gives

$$x(\theta)=\sinh^{-1}(\overline{b})+b\left(E\left(\theta+\tfrac{\pi}{4}\,\middle|\,\tfrac{2\,\overline{b}^2}{b^2}\right)+E\left(\tfrac{\pi}{4}-B(b)\,\middle|\,\tfrac{2\,\overline{b}^2}{b^2}\right)\right)+\overline{b}\,(\cos\theta+\sin\theta)+\frac{b\,\overline{b}}{\sqrt{1+\overline{b}^2}}\,.$$

Having the two pieces of the road, we can extend them using periodicity to give a single road function $(x(\theta),-r(\theta))$ that can be plotted parametrically. Note that we cannot solve $x=x(\theta)$ for θ to get the road in the form $y=f(x)$. This approach via a single elliptic integral is how the bridge designers obtained the proper shape for the road.

The use of a square for the bridge shape is somewhat unnatural. It works, but one might ask whether a circle would work just as well, as a circle can roll on a straight line. It can be done, but the track will take up more horizontal space than the track for the square. The wheel in Figure 19 turns upside down when rolled on its semicircular part and has a deck of 2 units and clearance of 2 units, for comparison with the square. The radius of the semicircle is $\sqrt{2}$ and the overall track length is $\pi\sqrt{2}+2$, or 6.44, compared to about 5.5 for the square. There are other ways to get an invertible bridge with a nicely short track, discussed in the next section.

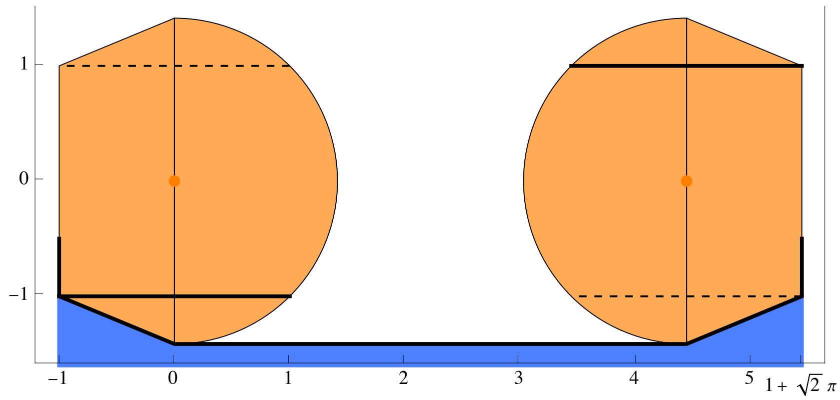

Figure 19. The rolling part of the bridge frame can be a semicircle, but the overall track length is larger than that for the square wheel.

7. Triangular Wheels

A triangle is even farther from a circle than a square, though the square is embedded in the public consciousness as the true opposite of a circle. But the triangle is extremely interesting from the rolling perspective. As discovered by Hall and Wagon [2], a triangle cannot physically roll on the appropriate catenaries because a corner crashes into the road before it strikes the cusp (Fig. 20, left). But, borrowing an idea from the Cody Dock bridge, we can get a rolling triangle if we round the corners. If the triangle side measures two units, then the minimal rounding radius that eliminates the collision is approximately 0.0831. Finding this value is an interesting problem in numerical optimization: Given a rounding radius, we check all possible positions of the wheel on the road to see whether a collision occurs. Repeating this procedure gets us the minimum rounding radius for which no collision occurs. Note that the triangle faces a steeper slope than the square does, so for a physical model one must use materials that prevent slippage when the contact point is at a steep spot on the road.

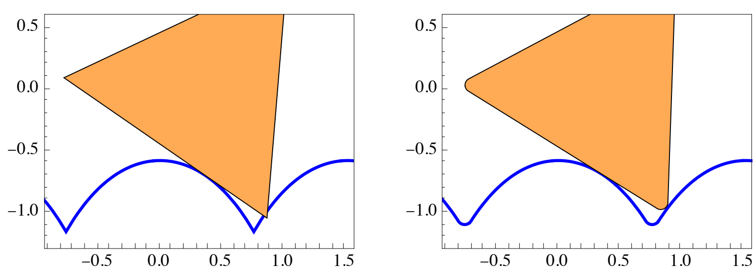

Figure 20. Left: A triangle cannot work as a wheel because there is a serious collision between a vertex and the catenary road. Right: Rounding the corners by 5% (the rounding radius as a percentage of the side length) eliminates the collision.

To get the equation of the road for the rounded triangle we start with the triangle centered at $(0,0)$. The start of the road will be part of a catenary. Then we need to accommodate the first rounded corner; after that we can just extend periodically. The corner is not easily expressed in polar form $r = r(\theta)$, because for a circle in a general position, the relation between r and θ is a quadratic equation in r. (The case of a square wheel with rounded corners is special, because the centers of the circles are on the lines $y = \pm x$ leading to considerable simplification.) So it is again easier to represent the circular corner as a parametric curve $g(t) = (g_1(t), g_2(t))$; the corresponding part of the road (see Fig. 20, right) is a part of the loopy curve discussed in §6.

The triangular frame does not work well for a bridge because the acute angles interfere with passage along the deck. But we can truncate the shape to get a pentagon that works nicely: its perimeter is less than that of the square and so it can be rolled upside-down in less space than the square requires; see Figure 21.

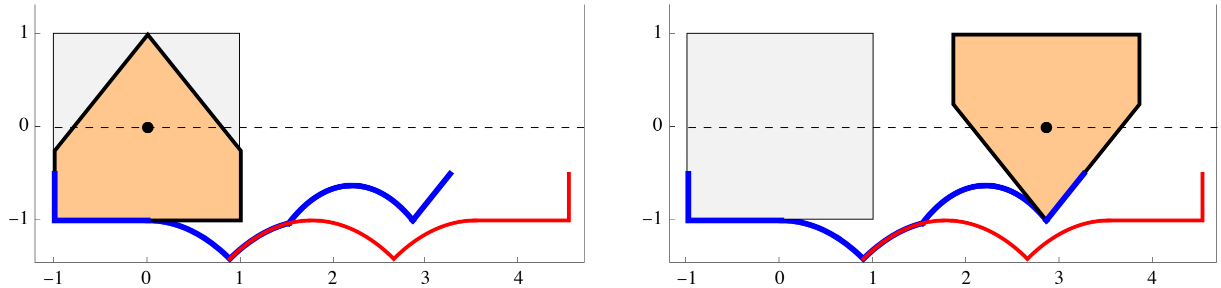

Figure 21. A pentagonal shape can be inverted using three catenaries (blue). The red curve shows the track for a square (shown in gray) with the same size bridge deck.

8. Conclusion: The Perfect Catenary

Catenaries have been used in bridge construction for a long time. An inverted catenary is the shape that forms the strongest arch, and so one sometimes sees such curves in the arches supporting a bridge from below. More common is a catenary arising as the shape of a cable holding a bridge from above, as the hanging-chain form is the definition of a catenary (from *catena*, Latin for *chain*). The Cody Dock bridge is a completely new use of catenaries in bridge construction, as the curve lies below and to the side of the bridge and allows the bridge to be turned upside-down. Moreover, the traditional appearance of catenaries is imperfect: for example, the weight of the bridge deck means that the cables for a bridge do not form exact catenaries. But the Cody Dock bridge uses a perfect catenary for the surface on which the straight part of the bridge rolls.

References

1. K. Bokhammas, Ready to roll, *Bridge Design and Engineering*, No. 109, 2022, 48–49.

2. L. Hall and S. Wagon, Roads and wheels, *Math. Mag.*, 65(5), 1992, 283–301.

3. J. C. Maxwell, On the theory of rolling curves, *Trans. Royal Soc. Edinb.*, 16(5), 1849, 519–540. http://archive.org/details/transactionsofro16roy/page/518/mode/2up .

4. Thomas Randall-Page, Cody Dock Rolling Bridge, https://thomasrandallpage.com/Cody-Dock-Rolling-Bridge .

5. Thomas Randall-Page, Cake Industries, Price & Myers, Cody Dock Rolling Bridge: Fabrication, https://www.youtube.com/watch?v=M_HTNX2M3X8 .

6. G. B. Robison, Rockers and rollers, *Math. Mag.*, 33(3), 1960, 139–144. www.jstor.org/stable/3029034?origin=crossref .

7. A. Slavik, S. Wagon, and D. Schwalbe, *VisualDSolve: Visualizing Differential Equations with Mathematica*, 2nd ed., Wolfram Research, Inc., 2009, Chap. 12, http://www.wolfram.com/books/profile.cgi?id=9553 .

8. S. Wagon, The ultimate flat tire, *Math Horizons*, (Feb. 1999) 14–17.

9. S. Wagon, Shaping a road and finding the corresponding wheel, Wolfram Demonstrations Project, 2011, http://demonstrations.wolfram.com/ShapingARoadAndFindingTheCorrespondingWheel

10. S. Wagon, A rolling square bridge: Reimagining the wheel, Wolfram Community, May 9, 2023, https://community.wolfram.com/groups/-/m/t/2917199

11. D. G. Wilson, Problem E1668, *Amer. Math. Monthly*, 71(2), 1964 p. 205; solution, 72(1), 1965, 82–83.

12. Exploratorium, Square wheels, https://www.exploratorium.edu/snacks/square-wheels .