Problem 1355. Square Root of Sine.

Find a twice differentiable function $f(x)$ defined for $-\pi/2 \leq x \leq \pi/2$ that is a compositional square root of sine: that is, $f(f(x)) = \sin x$ for all x in the domain.

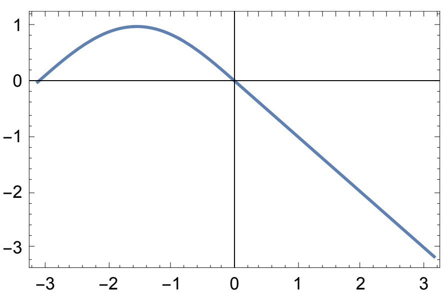
Source: T. Chen & E. Scheinerman (Nov. 2022) Finding a compositional square root of sine, *American Mathematical Monthly*, 129:9, 816-830. See also T. Curtright et al, Approximate solutions of functional equations, <https://arxiv.org/pdf/1105.3664.pdf>, and < https://oeis.org/A098932 >.

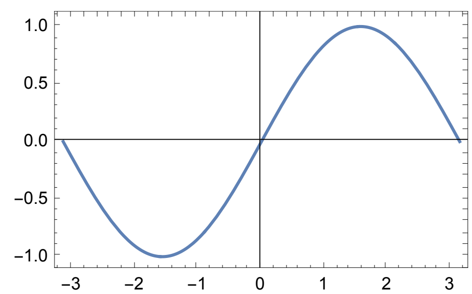
Solution.

Solved by David Broadhurst (UK), Barry Cox (Australia), Colin McGregor (Scotland), and partially by Dan Dima (Romania, via series). I should have asked for the interval $[-\pi, \pi]$ as the solution works there. The answer is:

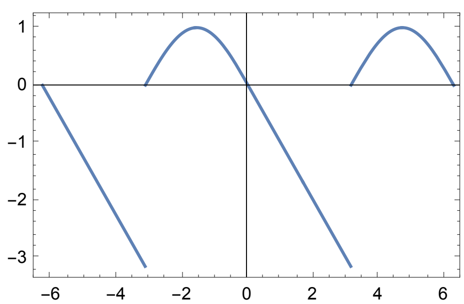
Let $f(x)$ be
$-x$ when $0 \leq x \leq \pi$
$-\sin x$ when $-\pi \leq x < 0$.

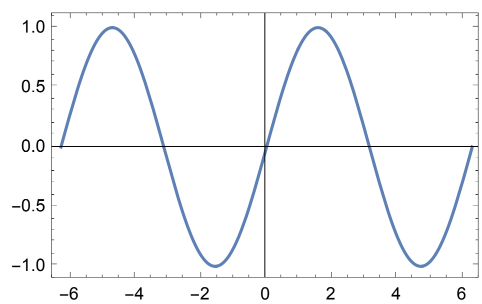
The graphs of f and $f \circ f$ (which equals sine) are:



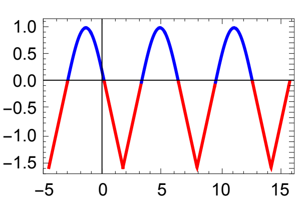


Jim Tilley observes that extending f periodically leads to a (discontinuous) solution on the entire real line; graphs below (the composition is at right).





However, if we flip each new segment before repeating it, then the result is continuous, and is again a compositional square root of π. One way of saying this is to use $f(\arcsin(\sin x))$. The graph is this:



The natural question is whether there is a differentiable, perhaps infinitely differentiable, solution on the real line.

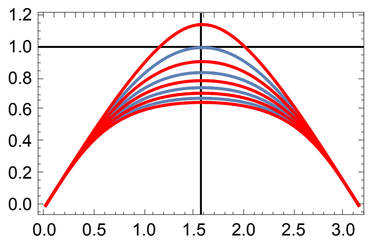
The source paper refers to notes of J. Écalle that deduces the existence of an infinitely differentiable solution from some very advanced ideas. One can find direct solutions that almost surely work, but proofs are lacking.

Conjecture 1 (Ed Scheinerman). The following function is an infinitely differentiable compositional square root of sine: Let $\arcsin^{(n)}$ denote the nth iterate of arcsine and let $\sin^{(n)}$ be the nth iterate of sine. Define

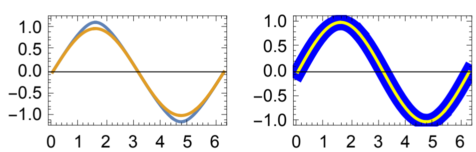
$$g(x) = \lim_{n\to\infty} \arcsin^{(n)}\left[\frac{1}{2}\left(\sin^{(n)} x + \sin^{(n+1)} x\right)\right]$$

Ed describes his motivation for this as follows: “I should note the intuition behind my iterated sine formula. What one can observe is that subsequent iterates of sine become roughly evenly spaced. That follows from the fact that $\frac{\sin x}{x} \to 1$ as $x \to 0$. Thus the “half sine” iterates should lie halfway between the sine iterates. So I iterate sine a bunch. Average with the next iterate of sine. And then arcsine my way back.“

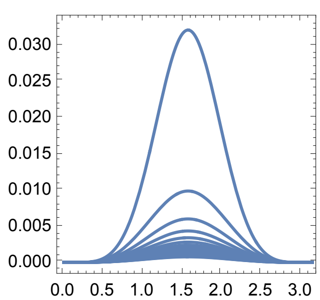
This graph shows Ed’s reasoning. The blue curves are five iterates of sine. The red graphs use the formula for g, with $n = 10$ in all cases, but with the number of arcsines being 0, 1, 2, or 3 less than the number of iterates of the sine in g.



Here is the graph of g (left, blue, with sine in orange; using $n = 20$ as an approximation to the limit). And on the right is the graph of $g \circ g$ in yellow with sine in blue. The graphs extend periodically and, for fixed n, are infinitely differentiable and analytic when n is fixed. It appears that the limit is infinitely differentiable. Ed S. did prove that the pointwise limit exists.



Here is a graph of the decreasing error (i.e., $|g(g(x)) - \sin x|$ as n rises to 100. A loglog plot of the behavior at $x = \frac{\pi}{2}$ indicates that the error is about $\frac{1}{n}$, and this provides good evidence that the error approaches 0 as $n \to \infty$.



A different approach (studied by contributors Dima, Cox, and Broadhurst) is to assume that a solution in the form of a series in powers of x exists and then solve for the coefficients. See the OEIS entry. The resulting hoped-for solution is

$$h(x) = x - \frac{1}{12}x^3 - \frac{1}{160}x^5 - \frac{53}{40\,320}x^7 - \frac{23}{71\,680}x^9 \ldots .$$

The coefficients arise from the sequence $1, -1, -3, -53, -1863, -92\,713, -3\,710\,155, 594\,673\,187, \ldots$, dividing by
$2^{n-1}(2n-1)!$. Here n refers to the index in the list; for example: $n = 2$ gives the cubic coefficient as $(-1)/(2^1\,3!)$, or $-\frac{1}{12}$. Note that the second through seventh are negative, but that does not last. Note also that the terms decrease for a while, but then increase and they go to infinity.

The first 100 coefficients of h can be found at the OEIS link. But PoW contributor David Broadhurst has computed the first 1000 terms. I posted the first 300 at

<stanwagon.com/public/1355Broadhurst300Coefficients.txt>.

Here is a plot of log10 of the absolute values of the coefficients. Broadhurst has observed that if one plots every fourth one (four plots), then the pattern is much smoother.

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ListLinePlot[Log10[Abs[aa /@ Range[1, 70]]], Frame → True, PlotRange → All]
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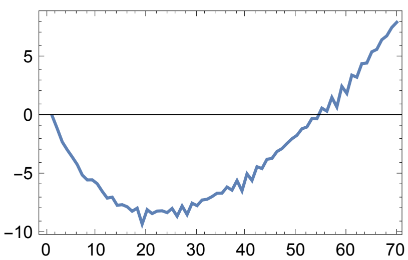

It turns out that the series $h(x)$ diverges for all nonzero values of x. One might think that this is a disaster, but the Curtright paper uses this idea: consider $\arcsin^{(n)}\left[h\left[\sin^{(n)} x\right]\right]$, where h is the power series; that is, conjugate h by sine n times. This leads to a periodic function and as $n \to \infty$ it appears to converge very quickly (exponential decrease in error) to a compositional square root. This function appears indistinguishable in the limit from the simpler example g above, but the convergence is faster. Here is a semilog plot (base 10) of the absolute error (at $x = \frac{\pi}{2}$), using Broadhurst's 300 coefficients. The linearity indicates exponential speed in the error reduction. The error is less than 10^{-350} when 300 terms and 300 iterates in the conjugation. Nice!

Conjecture 2. The sequence $\arcsin^{(n)}\left[h\left[\sin^{(n)} x\right]\right]$ converges as $n \to \infty$ to an infinitely differentiable compositional square root of sine.