Problem 1353. Bet on Red

You are facing a standard deck of 52 cards face down. You turn them over one at a time until you say STOP. Then you win if the next card is red. If you STOP before turning any card over, your chance of winning is exactly one half. If you do not step after seeing 51 cards, then you must say STOP before the last card. Is there a strategy for which your chance of winning is greater than one half?

Extra Credit. As above, but suppose you start with \$1 and can bet any amount (a real number between 0 and X, where X is your current stake) on the color of the next card. When you are correct, you win the amount that you bet (that is, you are playing at even odds). You go through all 52 cards. What is the maximum expected value of any strategy you can apply? (For example, if you never bet, your expected stake value is \$1. Or you can bet nothing until the last card. You will know its color, so you will win \$1 and the expected value of this strategy is \$2. There are better strategies.)

Source: These problems are folklore. A good reference is Winkler, Mathematical Puzzles, A Connoisseur's Collection, p. 67, 71-75.

Solution. The answer to Problem 1353 is that no strategy is any better than simply stopping before turning over the first card. This is a bit surprising, as one might think that by waiting until there are more reds than blacks remaining one could come up with a strategy that beats 1/2. This result was proved by: David Stigant and Vitor Schroeder dos Anjos,

For the second problem the optimal value is \$9.08, which was found by Mark Rickert and Vitor Schroeder dos Anjos. No one proved that this was optimal. The proof is tricky and is given below.

Here is the very slick proof for Problem 1353:

Suppose you use strategy S. Consider a variant to the game where the revealed card is the last card. When strategy S is applied to this last-card game, the chance of success is clearly $1/2$. But, for any position, the next card has the same probability of being red as does the last card: the probabilities are $r/(r+b)$, where r and b are the numbers of red and black cards in the deck following the specified position. To conclude, we show that the success probability for S is the same in both games.

Let $C(i)$ be the event that card i is red. Suppose x is a sequence of card colors, with length $|x|$. Call x a *trigger* if strategy S specifies that you stop after seeing x dealt. Every deck has exactly one initial segment that is a trigger: the sequence after which S calls for a stop. For a trigger x, let $I(x)$ be the event that the deck starts with x. Then the probability of winning is:

$\sum_{\text{triggers } x} P[I(x)] \cdot P[C(|x| + 1) \text{ given } I(x)]$

By the earlier remark about the variant of the game, this equals

$\sum_{\text{triggers } x} P[I(x)] \cdot P[C(52)] \text{ given } I(x)]$

Because the events I_x are disjoint and exhaustive, this is just $P[C_{52}]$, or $1/2$. □

Another approach uses induction on the size of the deck, proving the more general result that any strategy wins with probability $r/(b+r)$ where the deck starts with r reds and b blacks. If you guess immediately, your probability of success is $r/(b+r)$. If you don't, then use the induction hypothesis and consider the two cases: first card is black, first card is red. You are facing $b-1$ blacks and r reds, or b blacks and $r-1$ reds. Using the induction hypothesis, the probability of success then is:

$$\frac{b}{b+r}\ \frac{r}{b+r-1} + \frac{r}{b+r}\ \frac{r-1}{b+r-1}$$

which simplifies to $\frac{r}{b+r}$. □

EXTRA CREDIT PROBLEM

For the second problem, define a strategy to be "reasonable" provided that whenever the remaining cards all have the same color, the strategy bets the entire stake (might be 0) on that color at each bet. Clearly any optimal strategy must be reasonable. The surprising result here is that all reasonable strategies are equivalent. They all have expected value $2^{52} \Big/ \binom{52}{26}$, which is 9.08. Therefore \$9.08 is the maximum expectation.

An example of a reasonable strategy is to bet your entire stake on each card, making 26 red bets in a row, followed by 26 bets on black. You will go broke with probability 0.99999999999999798. But in the one case where the cards match your bets, you will end up with 2^{52} dollars. Therefore your expected return is $\frac{2^{52}}{\binom{52}{26}}$, or \$9.08.

Here is another reasonable strategy: If there are r red cards remaining and b black cards, your stake is X, and $r \geq b$, bet $X\ \frac{r-b}{b+r}$ that the next card is red; if $r < b$, bet $X\ \frac{b-r}{b+r}$ on black. The expected return is again: $\frac{2^{52}}{\binom{52}{26}}$. This can be proved by induction (details omitted).

The surprise is the following, which implies that either of the two aforementioned strategies, or any other reasonable strategy, is optimal.

Theorem. Any reasonable strategy has expected value $\frac{2^{52}}{\binom{52}{26}}$.

Proof. Let D be the number of possible decks: $D = \binom{52}{26} = 49\,591\,85\,329\,48\,104$, or half a quadrillion. Let $C = \frac{2^{52}}{D} = 9.08\ldots$. Recall from the first example above that each strategy that picks a deck and plays as if the random deck is the selected one has expected value C. Suppose you are playing according to some strategy S.

Imagine that you have D assistants, each one assigned to one of the possible decks. The assistants will bet according to the deck they are assigned to, betting their entire stake each time. The stakes for these assistants are defined from S as follows.

Consider a specific example. Suppose S starts by betting 8¢ on red. Divide the \$1.00 stake among the assistants as follows: 54¢ to the set of those assistants who will bet red and 46¢ to the others. Now consider the second step for S: There are four possibilities: RR, RB, BR, BB. Consider RR and RB. Suppose S would bet 4¢ on black in this situation. Then the 54¢ previously assigned to the red group is split into 29¢ and 25¢, with the 29 going to the RB assistants, and the 25¢ to the RR assistants. This idea is used through all 52 steps to go from S to an assignment of \$$A(i)$ to the ith assistant. So at the end, each assistant has his or her stake of \$$A(i)$. Let $T(i)$ be the strategy played by assistant i, assuming stake \$1 to start. Then, for each i, $E(T(i))$ is C.

Note that the partition of the stake runs into a problem (division by 0) if S is unreasonable. For example, suppose the deck has 2 cards (so only two decks: RB and BR) and strategy S says to bet \$0 on the first card being red and then \$1 on the second card being black. The first division assigns 50¢ to RB and 50¢ to BR. The second division will keep the 50¢ to RB, but the 50¢ assigned to BR must be divided by the number of recipients that will bet on black at round two. That number is 0 so it cannot be done.

Claim. S is a combination of the strategies $T(i)$: $S = A(1)\,T(1) + A(2)\,T(2) + \ldots + A(D)\,T(D)$.

Proof of Claim. The claim statement means that the result of any bet using S is the same as the result, at that stage of the game, of a combination of the $T(i)$ with coefficients $A(i)$. At the start, S bets \$$X$ on red. But that is what happens in the convex combination: The players whose selected deck starts with red will, jointly, bet \$$X$ on red. This works for all 52 steps.

Claim. If S is a reasonable strategy, then the combination of the preceding claim is a convex combination, meaning that $\Sigma_i A(i) = 1$.

Proof. If S is reasonable, then, as the stake is divided up, it will always be possible to make the division according to the division rule given earlier: if there are both R and B left in the deck, then the division is possible. If there is only one color left, then again the division is possible because S, being reasonable, will bet on the remaining color. □

Because expectation is a linear function over convex combinations, the claims mean that

$E(S) = \sum A(i)\, E[T(i)] = \sum A(i)\, C = C$. □

A relevant source is equation 15 at:

< https://purl.stanford.edu/js411qm9805 >