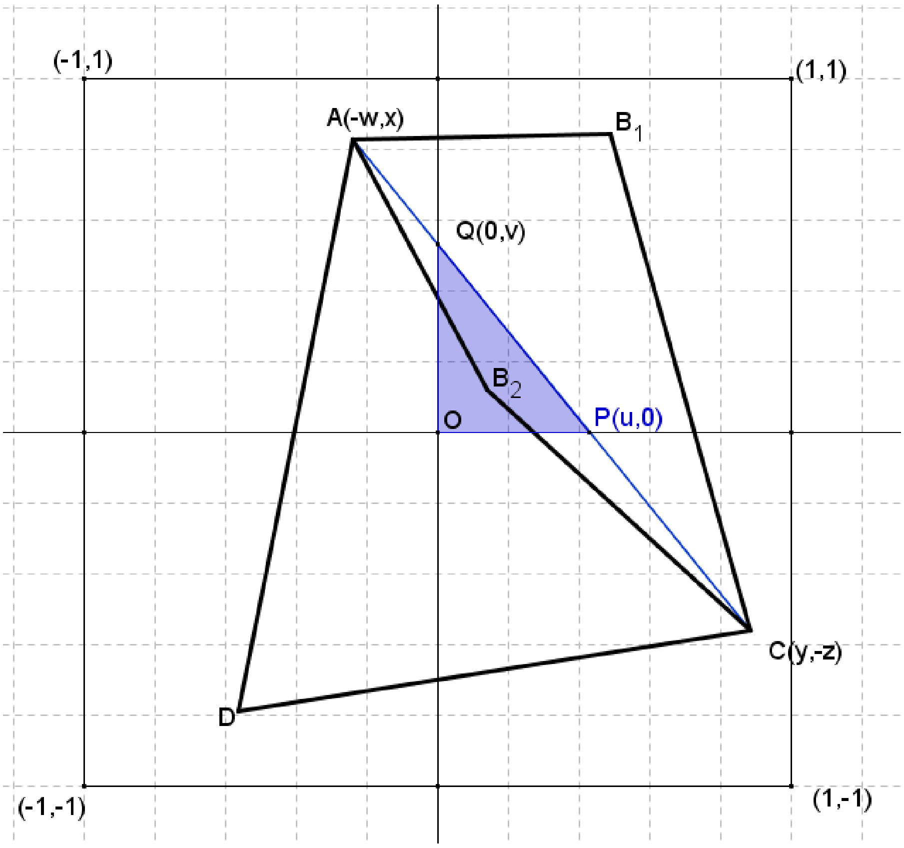
Problem # 1305 Convex probability

Choose four points in at random in the square with vertices (1,1), (-1,1), (-1,-1), (1,-1), one in each quadrant. What is the probability that they form a convex quadrilateral?

Source: Oliver Roeder's problem book, "The Riddler", Norton & Co., 2018, p. 179. Roeder runs a problem section for <FiveThirtyEight.com>; see <https://fivethirtyeight.com/tag/the-riddler/>

Solution proposed by Philippe Fondanaiche

Answer : the probability that the quadrilateral is convex is close to 91%



Let consider a quadrilateral ABCD whose vertices are chosen respectively in each quadrant (-1,1),(1,1),(1,-1),(-1,-1). This quadrilateral is convex if and only if the four internal angles $\angle ABC$, $\angle BCD$, $\angle CDA$, $\angle DAB$ are less than 180 degrees. In the above figure see the quadrilateral AB_1CD.

It becomes nonconvex if one of these angles is greater than 180 degrees. In the figure see the quadrilateral AB_2CD with $\angle AB_2C > 180°$. The corresponding vertex is called « re-entrant ».

Lemma n°1: a quadrilateral is nonconvex with at most one re-entrant vertex

The sum of the internal angles of a quadrilateral is equal to 360°. If one these angles is greater than 180°, another one cannot be greater than 180° as the sum of the two angles would be >360°. Contradiction.

Lemma n°2 : **a quadrilateral is nonconvex if one of the vertices is chosen in a right triangle based on the abscissa-axis and the ordinate-axis.**

See the above figure with the point B_2 located in the triangle OPQ with P(u,0) and Q(0,v) at the intersections of the segment AC with the abscissa-axis and the ordinate-axis

Let p = probability of {ABCD = convex quadrilateral }

p = 1 – probability of {ABCD = nonconvex quadrilateral}.

Probability of {ABCD = nonconvex quadrilateral} = Probability of a point located in the quadrant (-1,1) or in the quadrant (1,1) or in the quadrant (1,-1) or in the quadrant (-1,-1)

By symmetry the probabilities of a point located in each of the quadrants are the same and are equal to area triangle OPQ/ area square(1*1) = area of triangle OPQ.

According to the lemma n°1,the four events “located in quadrant (i,j)” i = –1,1 and j=–1,1 are disjoint.

So p = 1 – 4 area OPQ = 1 – 2uv.

Let the coordinates A(-w,x) and C(y,-z) with **w,x,y,z uniformly and independently distributed over the intervals [0,1].**

We get u = (xy – wz)/(x + z) and v = (xy – wz)/(w + y).

So to find p, we integrate uv over 0≤ w,x,y,z ≤ 1 without forgetting to divide the result by 2 in order to take into account the case where the variables u and v are both negative.

Therefore $p = 1 - \iint uv\,du\,dv = 1 - \int_0^1\int_0^1\int_0^1\int_0^1 \frac{(xy-wz)^2}{(w+y)(x+z)}\,dw\,dx\,dy\,dz$

Thanks to the WolframAlpha software (see appendix) , we get :

p = 11/6 – 4log2/3 ≈ 0.9091…

APPENDIX

WolframAlpha computational intelligence.

integrate (xy-wz)^2/((w+y)*(x+z))dwdxdydz from w=0 to 1 x=0 to 1 y=0 to 1 z = 0 to 1

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Definite integral:

$$\int_0^1\int_0^1\int_0^1\int_0^1 \frac{(x\,y-w\,z)^2}{(w+y)(x+z)}\,dw\,dx\,dy\,dz = 0.0908629$$

integrate (xy-wz)^2/((w+y)*(x+z))dwdxdydz from w=0 to 1

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Definite integral:

$$\int_0^1 \frac{(x\,y-w\,z)^2}{(w+y)(x+z)}\,dw = y^2\,(x+z)\,(\log(y+1)-\log(y)) + \frac{z\,(z-2\,y\,(2\,x+z))}{2\,(x+z)}$$

for $\mathrm{Re}(y) > 0 \lor \mathrm{Re}(y) < -1 \lor y \notin \mathbb{R}$

integrate y^2*(x+z)(log(y+1)-log(y))+z(z-2*y*(2x+z))/(2(x+z))dx from x=0 to 1

Definite integral:

$$\int_0^1\left(y^2\,(x+z)\,(\log(y+1)-\log(y)) + \frac{z\,(z-2\,y\,(2\,x+z))}{2\,(x+z)}\right)dx =$$
$$y^2\left(z+\frac{1}{2}\right)(\log(y+1)-\log(y)) + \frac{1}{2}\,z\,((2\,y+1)\,z\,(\log(z+1)-\log(z))-4\,y)$$

for $\mathrm{Re}(z) > 0 \lor \mathrm{Re}(z) < -1 \lor z \notin \mathbb{R}$

integrate y^2(z+1/2)(log(y+1)-log(y))+(z/2)((2y+1)z(log(z+1)-log(z))-4y)dy from y=0 to 1

Definite integral: Step-by-step solution

$$\int_0^1\left(y^2\left(z+\frac{1}{2}\right)(\log(y+1)-\log(y)) + \frac{1}{2}\,z\,((2\,y+1)\,z\,(\log(z+1)-\log(z))-4\,y)\right)dy =$$
$$\frac{1}{12}\left(12\,z^2\,\log(z+1)-14\,z+z\,\log(256)-1+\log(16)\right)-z^2\,\log(z)$$

integrate (1/12)*(12z^2log(z+1)-14z+zlog(256)-1+log(16))-z^2log(z) dz from z=0 to 1

Definite integral: More digits Step-by-step solution

$$\int_0^1\left(\frac{1}{12}\left(12\,z^2\,\log(z+1)-14\,z+z\,\log(256)-1+\log(16)\right)-z^2\,\log(z)\right)dz =$$
$$\frac{1}{6}(\log(256)-5) \approx 0.090863$$