A Historically Interesting Truth-Teller Problem

Suppose Alice, Bob, Charlie, and Diane tell the truth with (independent) probability 1/3. Alice stated that Bob denied that Charlie declared that Diane lied. What is the probability that Diane told the truth?

Details. We assume that Diane made a certain statement whose truth value both Charlie and Diane knew (but of course Diane lied about it with probability 2/3). Then Charlie said one of "Diane told the truth" or "Diane lied" according to what he heard and his own probability of lying. Bob heard what Charlie said, and he said one of "Charlie said that Diane told the truth" or "Charlie said that Diane lied." And Alice heard Bob's statement and made her assertion accordingly.

Solution: The answer is 13/41. To see why, first note that for Bob to deny that Charlie declared that Diane lied, Bob must assert that Charlie said that Diane told the truth. Thus, the given information must mean that Alice said that Bob said that Charlie said that Diane told the truth.

A crucial observation is that Alice would say this if and only if an even number of the participants lied. For example, suppose Alice and Charlie told the truth but Bob and Diane lied. Then Charlie must have truthfully said that Diane lied, but Bob lied and claimed that Charlie said that Diane told the truth. Alice would then have told the truth and said that Bob said that Charlie said that Diane told the truth. On the other hand, if only Bob lied, then Diane told the truth, Charlie reported this truthfully, Bob lied and claimed that Charlie said that Diane lied, and then Alice truthfully said that Bob said that Charlie said that Diane lied.

Thus the question can be rephrased as asking for the probability that Diane told the truth, given that an even number of people lied. Let E be the event that an even number people lied, and let D be the event that Diane told the truth. For the event E to occur, the number of liars must be 0, 2, or 4. Therefore

$$P(E) = \left(\frac{1}{3}\right)^4 + \binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^4 = \frac{41}{81}.$$

On the other hand, for both D and E to occur Diane must tell the truth, and the number of liars among the others must be either 0 or 2. Therefore

$$P(D \wedge E) = \left(\frac{1}{3}\right)^4 + \binom{3}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 = \frac{13}{81}.$$

Therefore the requested probability is

$$P(D \mid E) = \frac{P(D \wedge E)}{P(E)} = \frac{13/81}{41/81} = \frac{13}{41}.$$

Now we generalize. Suppose there are n people, $P_1, P_2, \ldots, P_n$, all of whom tell the truth with (independent) probability p, where $0 < p < 1$. Person P_n makes a statement whose truth value is known to both P_n and P_{n-1}. Then P_{n-1} says either "P_n told the truth" or P_n

lied," P_{n-2} says either "P_{n-1} said that P_n told the truth" or "P_{n-1} said that P_n lied," and in general, for $1 \le i \le n-1$, P_i says either "P_{i+1} said that P_{i+2} said that … P_{n-1} said that P_n told the truth" or "P_{i+1} said that P_{i+2} said that … P_{n-1} said that P_n lied." If P_1 says that P_2 said that … P_{n-1} said that P_n told the truth, what is the probability that P_n told the truth?

As before, the question is equivalent to asking for the probability that P_n told the truth, given that there were an even number of liars. Let e_n be the probability that, among n people who independent tell the truth with probability p, the number of liars is even, and let o_n be the probability that the number is odd. Then

$$e_n = \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} p^{n-k}(1-p)^k, \qquad o_n = \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k} p^{n-k}(1-p)^k.$$

Of course, the number of liars must be either even or odd, so $e_n + o_n = 1$. Also, notice that

$$\begin{aligned} e_n - o_n &= \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} p^{n-k}(1-p)^k - \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k} p^{n-k}(1-p)^k \\ &= \sum_{k=0}^{n} \binom{n}{k} p^{n-k}(p-1)^k = (2p-1)^n, \end{aligned}$$

where in the last step we have used the binomial theorem. Combining these two equations, we find that

$$e_n = \frac{(e_n + o_n) + (e_n - o_n)}{2} = \frac{1 + (2p-1)^n}{2}.$$

Let E_n be the event that there are an even number of liars among $P_1, \ldots, P_n$, and let T_n the event that P_n told the truth. Then for the event $E_n \wedge T_n$ to occur, P_n must tell the truth and an even number of the other $n-1$ must lie, so we have

$$P(E_n) = e_n, \qquad P(T_n \wedge E_n) = p \cdot e_{n-1}.$$

Therefore the requested conditional probability is

$$P(T_n \mid E_n) = \frac{P(T_n \wedge E_n)}{P(E_n)} = \frac{p \cdot e_{n-1}}{e_n} = p \cdot \frac{1 + (2p-1)^{n-1}}{1 + (2p-1)^n}.$$

It is interesting to investigate how this answer changes as n increases. We have $0 < p < 1$, and therefore $-1 < 2p - 1 < 1$ and $(2p-1)^n \to 0$ as $n \to \infty$. Thus,

$$\lim_{n\to\infty} P(T_n \mid E_n) = \lim_{n\to\infty} p \cdot \frac{1 + (2p-1)^{n-1}}{1 + (2p-1)^n} = p \cdot 1 = p.$$

In other words, when the information about P_n's truthfulness is filtered through a long string of equally unreliable reporters, that information has almost no effect on the probability that P_n told the truth.

We can examine this phenomenon more closely by considering three cases. If $p = 1/2$ then $2p - 1 = 0$, and it follows that for all n, $P(T_n \mid E_n) = 1/2 = p$. Thus, the probability that P_n was truthful is completely unaffected by a claim of P_n's truthfulness that is filtered through reporters who tell the truth and lie equally often.

For $p \neq 1/2$, we can investigate the discrepancy between $P(T_n \mid E_n)$ and p by computing

$$P(T_n \mid E_n) - p = p \cdot \frac{(2p-1)^{n-1} - (2p-1)^n}{1 + (2p-1)^n} = \frac{2p(1-p)/(2p-1)}{1 + (1/(2p-1))^n}.$$

If $1/2 < p < 1$ then this quantity is positive but decreases monotonically to 0 as $n \to \infty$. This means that if the claim that P_n told the truth comes through a string of reporters who tell the truth more often than they lie, then it increases the probability that P_n told the truth, but the effect decreases to 0 as the length of the string of reporters increases.

Finally, consider the case in which $0 < p < 1/2$. In this case, $P(T_n \mid E_n) - p$ is positive if n is odd and negative if n is even. Thus, when the information of P_n's truthfulness comes through a string of reporters who lie more often than they tell the truth, our confidence that P_n told the truth will increase or decrease depending on the parity of the length of the string. Once again, in both the even case and the odd case, the strength of the effect decreases to 0 as n increases.

This problem arose from the famous May 1919 eclipse expeditions that tried to measure whether starlight bends as it passes near the sun. The observations were made by Sir Arthur Eddington and Edwin Turner Cottingham, on the island of Principe off the west coast of Africa, and Andrew Claude de la Cherois Crommelin and Charles Rundle Davidson, in Sobral, Brazil. According to Daniel Kennefick [D], Crommelin stated the truth-teller probability problem in an after-dinner speech before the expedition, but he used C, C', D, and E (presumably Crommelin, Cottingham, Davidson, and Eddington) instead of Alice, Bob, Charlie, and Diane. The problem was then discussed by Eddington in [E], where he gave a different answer from the one we have given, based on a different interpretation of the problem. See [D] for more on the history of the problem and the variations in interpreting it, and [K] for a detailed study of the experiment.

M. Deakin, A new look at Eddington's liar problem, Math Gazette 93 (no. 526) (2009), 1-6

D. Kennefick, No Shadow of a Doubt: The 1919 Eclipse That Confirmed Einstein's Theory of Relativity, Princeton NJ, Princeton Univ. Press., 2019.

A. Eddington, New Pathways in Science, Cambridge: Cambridge Univ. Press, 1935.