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1233 The Generous Automated Teller Machine. This was based on a 2010 Olympiad problem:

You have five boxes, B(1), B(2), B(3), B(4), B(5) and each one contains one coin. You may make moves of the following sort:

Type 1. Remove a coin from a nonempty B(i) and place two coins in B(i+1) (here
i is 1, 2, 3, or 4).
Type 2. Remove a coin from a nonempty B(i) and switch the contents of B(i+1) and B(i+2) (here i is 1, 2, or 3).

What is the largest number of coins you can place in B(5)? (or the extension to 6 boxes).

Solution. Consider the general problem of $n$ boxes $B(1),...,B(n)$ with arbitrary numbers of coins in them.

Call a Type 1 move an increment and a Type 2 move a flip. Call a box a flipper if the next move mediated by a coin in that box is a flip and call it an incrementer otherwise. First a few trivial observations:

Observation 0. Moves mediated by boxes $k,k+1,...,n$ can only effect boxes $k,k+1,...,n$ and the contents of $B(k)$ can only decrease (or stay unchanged).

Observation 1. A position which strictly dominates another (that is, in every box it has at least as many coins and in at least one box it has more) is always preferable.

Observation 2. An increment move mediated by box $k$ can always be commuted earlier through moves mediated by boxes $k+1,...,n$.
Proof. The increment move puts $2$ extra coins in box $k+1$. Since moves mediated by boxes $k+1,...,n$ can only decrement that box, there is no loss to putting them in earlier.

Observation 3. An flip move mediated by box $k$ can only occur when box $k+1$ is empty.
Proof. Suppose the contents of boxes $k+1$, $k+2$ are $a,b$ with $a>0$. Then a flip move would leave them $b, a$. However doing $a$ increments on $k+1$ first,
then the flip, then $a$ more increments on $k+1$ would send us from $a,b$ to $0,2a+b$ to $2a+b,0$ to $a+b,2a$. This is strictly better by Observation 1.

Now suppose box $k$ has the following properties

(1) Boxes $1,\dots,k-1$ are non-empty flippers, and
(2) Box $k$ is non-empty.

By Observation 3, no move mediated by boxed $1,\dots,k-1$ can occur until after the move by box $k$. Call such a $k$ a first responder.

Observation 4. If a first responder is an incrementer, then without loss that is the next move.
Proof. This is just Observation 2 and the remark above.

Observation 5. If a first responder is a flipper, and the next box is empty, then without loss it is the next move.
Proof. Since the next box, say $k+1$, is empty, any prior moves must be by boxes $k+2,\dots,n$. This can only lower the number of coins in box $k+2$. Hence by Observation 1, it is better to do the flip before it is lowered.

The remarkable thing is that one can ALMOST determine whether a first responder is a flipper or an incrementer and thus there is very little branching in the analysis.

Observation 6. Suppose box $k$ is a first responder and the contents of boxes $k+1$ and $k+2$ are $a,b$ coins, respectively.
  (i) If $2a+b\ge 4$, then without loss box $k$ is a flipper.
  (ii) If $a=0$ and $b\le 2$, then $k$ is an incrementer.

Proof. (i) If $2a+b\ge 4$ and $k$ is a incrementer , then by Observation 4 we may assume it is the next move, giving us $a+2,b$. If we used did $a$ increments from $k+1$, then used it to flip instead, we would get to $0,2a+b$ and then $2a+b,0$. Doing $a+b-2$ (which is non-negative since $2a+b\ge 4$) more increments from box $k+1$ would give $a+2, 2a+2b-4$. This is better if $2a+b>4$ and the same if $2a+b=4$.

(ii) If $k$ were a flipper, then by Observation 5 that flip might as well be the next move giving us $b,0$. However if we use it as an incrementer we get $2,b$ which is strictly better.

Thus the only cases where we cannot at a glance tell whether a first responder is a flipper or an incrementer are when the next two boxes are 03, 10, or 11.
We can do a little better in one important case.

Observation 7. Suppose $k$ is a first responder and all subsequent boxes have $1$ coin in them. Then $k$ is an incrementer.

Proof. If $k$ is a flipper, then $k+1$ is also a first responder and faces the same situation. Hence there is some least $m>k$ which is a first responder and an incrementer (since box $n-1$ must be an incrementer if we get that far). Then without loss this is the next move and we get boxes $m, m+1, m+2$ having $0, 3, 1$ coins, respectively. Now box $m-1\ge k$ is a flipper with a $0$ in the next box, so by Observation 5, it is without loss the next move giving 3, 0, 1. However if we just incremented from box $m-1$ we would have $3,1,1$ a preferred outcome.

These observations leave very few options for branching and one can without branching compute the optimum for 5 boxes. Here are a few steps, an $F$ labels a box known by Observation 6(i) to be a flipper, a number on an arrow labels the Observation being invoked.

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              7                7                 7
(1,1,1,1,1) --> (0,3,1,1,1) --> (0,2F,3,1,1) --> (0,2F,2F,3,1) -(3 steps)->

                5                                              5
(0,2F,2F,0,7) --> (0,2F,1F,7,0) -(7 steps)-> (0,2F,1F,0,14) --> (0,2F,0,14,0)

 5                 6
--> (0,1F,14,0,0) -->  (0,1F,13F,2,0) -(2 steps)-> (0,1F,13F,0,4)

-(cascade of flips and increments from box  4) -> (0,1F,1F,0,2^{14})
    forced since box  4  can only increment

 5                     5                     6
--> (0,1F,0,2^{14},0) --> (0,0,2^{14},0,0) --> (0,0,2^{14}-1 F,2,0)

-(cascade of flips and increments from box  4) --> (0,0,0,0,2^{2^{14}+1})
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For 6 boxes, the same works but given the length it is worth noting a "super-cascade" (SC below). If we have aF bF cF d 0 0 where $d\ge 2$ then we will after a cascade of cascades wind up at aF bF cF 0 $2^d$ 0 and then at aF bF (c-1)F $2^d$ 0 0. Then for 6 boxes we have

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                7                  7                   7                    7
(1,1,1,1,1,1) --> (0,3,1,1,1,1) --> (0,2F,3,1,1,1) --> (0,2F,2F,3,1,1) -->

                                       5
(0,2F,2F,2F,3,1)-->(0,2F,2F,2F,0,7) --> (0,2F,2F,1F,7,0) -->
(0,2F,2F,1F,0,14)

 5                    5
--> (0,2F,2F,0,14,0) --> (0,2F,1F,14,0,0) -(SC) -> (0,2F,0,2^{14},0,0)

 5                          6
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--> (0,1F,2^{14},0,0,0) --> (0,1F,2^{14}-1 F,2,0,0) -(SC)->

                                                                                5
(0,1F,2^{14}-2 F,4,0,0) -(sequence of SCs)-> (0,1F,0,2^^(2^{14}),0,0) -->

                          6
(0,0,2^^(2^{14}),0,0,0) --> (0,0,2^^(2^{14})-1 F,2,0,0) -(SC)->

(0,0,2^^(2^{14})-2 F,4,0,0) -(sequence of SCs) ->
(0,0,0,2^^(2^^(2^{14})),0,0)

--> (0,0,0,2^^(2^^(2^{14}))-1 F,2,0) -(cascade)->

(0,0,0,0,0,2^{1+2^^(2^^(2^{14}))}).
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