Solution to PoW 1205   Stephen Meskin, Univ of Maryland Baltimore County

Let the vertices of a triangle be A, B, and C and the lengths of the sides opposite A, B, and C be 2a, 2b, and 2c respectively. Then with three applications of the Pythagorean Theorem we can show that the square of the length of the median from A is 2b^2 + 2c^2 – a^2 and similarly for medians from B and from C. The medians intersect 2/3 of the way from the vertex to the opposite side.

Putting all this together we see that the squares of the lengths of the sides of triangle T1+T2+T3 are:

a^2   2b^2 + 2c^2 – a^2   4b^2

And the squares of the lengths of the sides of triangle T1 are:

(2b^2 + 2c^2 – a^2)/9   a^2   (2a^2 + 2b^2 – c^2)4/9

Where T1 sits on the side of length 2a,

Then T1~ T1+T2+T3 ➔ a^4 = ((2b^2 + 2c^2 – a^2)^2)/9 ➔

3a^2 = 2b^2 + 2c^2 – a^2 ➔

2a^2 = b^2 + c^2 which is the main equation.

Plugging the main equation into the 2nd item in the 1st line & the 1st and 3rd items in the 2nd line the two triplets become

a^2   3a^2   4b^2

(a^2)/3   a^2   (4b^2)/3

So the main equation is equivalent T1 ~ T1+T2+T3. The main equation is also equivalent to the other 5 similarities and the constants of similarity are all $\sqrt{3}$. The similarities can be permuted by letting T1 sit on the other two sides giving 12 new similarities.

We started with sides 2a, 2b, and 2c but if we divide everything by 2, the similarities remain and so does the main equation. So all we need to do to finish is to find 3 integers a, b, and c with minimal sum which forms a non-trivial triangle and satisfies the main equation. This can be done by a manual hunt.

It is possible to shorten the hunt by observing from the main equation that a must be between b and c. So we can set one of b & c equal to a–x and the other to a+y for some integers x and y. Then we must have x^2 +y^2 = 2a(x–y). Clearly x and y must both be odd or both even, But if they both odd then the LHS =2 mod 4 while the RHS =0 mod 4 so they are both even, moreover x>y>0. So now we can restrict our hunt to (x, y) =

(4, 2) ➔ a=5, with b & c = 1 & 7, which is not a triangle;

(6, 2) ➔ a=5 with b & c = -1 & 7, which again is not a triangle; and

(6, 4) ➔ a=13 with b & c = 7 & 17 which is a triangle and we are done.

(8, 2) ➔ a not integral

(8, 4) ➔ 2x(4,2)

(8, 6) ➔ a=25 with b & c = 17 & 33. Much bigger than (7, 13, 17) which appears to be the answer because other potential answers are likely to have perimeters greater than 37.