Problem 1200 was solved by David Broadhurst, Michael Elgersma, John Sullivan, and Witold Jarnicki. Here is the solution by Michael Elgersma, Minneapolis.

Given wind speed $\| w \|$, circular ground-track radius r, vehicle airspeed $\| v \| > \| w \|$, and time T to fly one circle, find an equation that relates the four variables; then solve for T in terms of the others.

The vehicle velocity vector u over the circular ground track is given by the following two equations, where $\theta(t)$ is the angle locating the position of the vehicle on the track. (1) follows from the fact that the airplane's postion is $r(\cos\theta(t),\ \sin\theta(t))$.

(1) $u = r\,\theta'\,(-\sin\theta, \cos\theta)$

(2) $u = w + v$ (see diagram)

The Law of Cosines gives (3) $\| v \|^2 = \| u \|^2 + \| w \|^2 - 2\,\| u \|\,\| w \|\,\cos\theta$

Using (1) in (3): (4) $\| v \|^2 = (r\,\theta')^2 + \| w \|^2 - 2\,r\,\theta'\,\| w \|\,\cos\theta$

Solving (4) for r θ' (5) $r\,\theta' = \| w \|\,\cos\theta + \sqrt{\| v \|^2 - \| w \|^2 \sin^2\theta}$

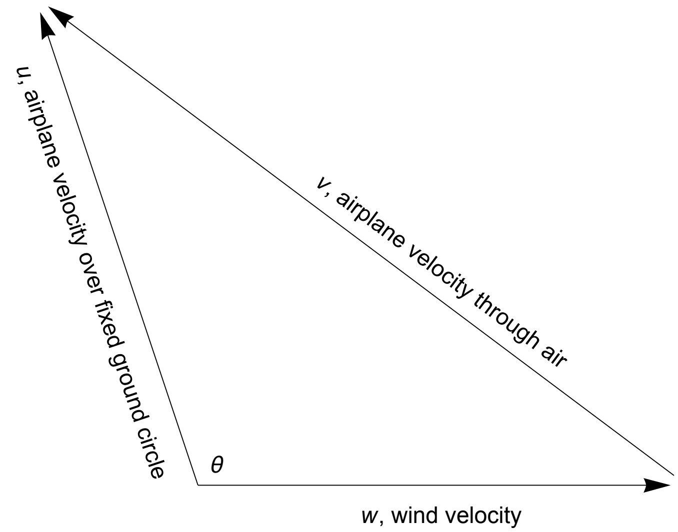
Integrating (5) and solving for T gives:

$$T = \int_0^T dt = \frac{r}{\|v\|}\int_0^{2\pi}\left(\frac{\|w\|}{\|v\|}\cos\theta + \sqrt{1-\left(\frac{\|w\|}{\|v\|}\right)^2 \sin^2\theta}\right)^{-1} d\theta$$

This last expression is an elliptic integral and can be expressed as follows (eliminating $\|\ \|$):

$T = \frac{4\,r\,v}{v^2-w^2}\,E\left(\frac{w^2}{v^2}\right)$

where E is the elliptic integral of the second kind (`EllipticE` in *Mathematica*). Such functions can be evaluated very quickly using ideas related to the arithmetic-geometric mean. See, e.g., <http://www.ams.org/notices/201208/rtx120801094p.pdf>.



So to solve the problem given just use basic root-finding. In *Mathematica*, using 1/2 hour as the time.

$$\texttt{r = 10; v = 145; FindRoot}\left[\frac{1}{2} == \frac{4\,r\,v\,\texttt{EllipticE}\left[\frac{w^2}{v^2}\right]}{v^2 - w^2}, \{w, 1\}\right]$$

$\{w \to 60.1167\}$

Now `FindRoot` has some overhead and precision control that is generally a good thing, but that can be eliminated. The *Mathematica* code below implements Newton's method, with some algebra used to simplify the Newton iterate, F. Precision is bootstrapped: ie., it is doubled at each iteration. Newton's method is iterated 18 times. This is complete code to get a million digits in 26 secs. One should compare the iterates to gain evidence for correctness. David Broadhurst, using PARI, used a direct AGM approach to calculate the two elliptic integrals simultaneously, and also tried the quartic Householder method. He got a million digits in under ten seconds.

$$\texttt{F[z\_] :=}\left(r = \frac{z^2}{21025};\ q = z^2 - 21025;\ t = \frac{q}{\texttt{EllipticE[r]}};\ z\left(1 + \frac{q\left(\frac{t}{11600} + 1\right)}{z^2 + 21025 + \texttt{EllipticK[r]}\,t}\right)\right);$$

$$\texttt{p = 2; Nest}\left[\left(p = \texttt{Min}\left[10^6 + 5, 2\,p + 2\right];\ s = \texttt{SetPrecision[\#, p]};\ \texttt{F[s]}\right)\texttt{\&, 60., 18}\right]$$

Following Broadhurst's advice and using an AGM approach to `EllipticK` as follows knocks the time down to 16 seconds for a million digits.

$$\texttt{EllipticKAGM[m\_] :=}\ \pi \Big/ \left(2\,\texttt{ArithmeticGeometricMean}\left[1, \sqrt{1 - m}\right]\right)$$

The million-digit result is as follows where the total number of digits is one million.

60.1166823751704163098778297543....61614635765